Odd linking

A new view on Min-Max

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Outline

1. The starting point: a Schrödinger equation
2. Definiteness— Indefiniteness
   - Below the spectrum
   - Inside a gap
3. The case where $\lambda < \lambda_1$
4. The case where λ is in a gap

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Non-linear Schrödinger equation of the form

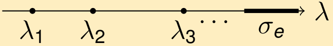
$$\begin{cases} -\Delta u(x) + V(x)u(x) - q(x)|u(x)|^\sigma u(x) = \lambda u(x), & x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \end{cases}$$

where

$N \geq 2$
$0 < \sigma < \frac{4}{N-2}$

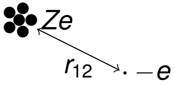
$q \in L^\infty(\mathbb{R}^N)$
$q > 0$
(to keep the presentation simple)

V a Coulomb-like potential, f.ex. $V(x) = -2/|x|$.



Spectrum of $-\Delta + V$: the hydorgen-like case

A nucleus of mass m_1 and of charge Ze is surrounded by an electron of mass m_2 and of charge $-e$:



Discrete spectrum:

$$\sigma_d(-\Delta + V) = \{\lambda_n \mid n \in L\},$$

where

- $\lambda_1 < \lambda_2 < \cdots < 0$, λ_1 being simple;
- $L = \mathbb{N}$ or $L = \{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$;
- $\lim_{n\to\infty} \lambda_n = 0^-$ if $L = \mathbb{N}$.

followed by a continuous spectrum $[0, \infty[$.

The formulation in an abstract setting

$$Au - \nabla\Phi(u) = \lambda Lu, \qquad u \in \text{ a Hilbert-space } H$$

with

- A and $L : H \to H$ are self-adjoint and bounded operators with

$$(Au, u) = \int_{\mathbb{R}} \nabla u \cdot \nabla V + Vuv\ dx$$

  and

$$(Lu, v) = \int_{\mathbb{R}} uv\ dx.$$

- $$\Phi(u) := \frac{1}{2+\sigma} \int_{\mathbb{R}} q(x)|u|^{2+\sigma}\ dx.$$

The corresponding energy functional

- Energy

$$I_\lambda(u) := \frac{1}{2}((A - \lambda L)u, u) - \Phi(u)$$

- "Weak" problem: for each $\lambda \in \sigma(-\Delta + V) = \sigma(A)$, find $u \in H \setminus \{0\}$ with

$$\nabla I_\lambda(u) = 0.$$

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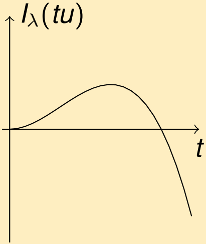
For $\lambda < \lambda_1$

The quadratic part of the energy is positive definite. Thus we have the following radial behavior:



For $\lambda < \lambda_1$

Thus I_λ is positive in a vicinity of 0, this being so uniformly:

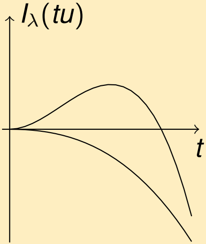
Proposition

We have, for $\lambda < \lambda_1$ kept fixed,that there exists $\rho_\lambda > 0$ and $\alpha_\lambda > 0$ such that

$$I_\lambda(u) \geq \alpha_\lambda > 0, \qquad \forall u \in H \text{ with } \|u\| = \rho_\lambda.$$

Inside a gap

The quadratic part of the energy is indefinite. Thus we have the following radial behavior:



Decomposition of H

Orthogonal decomposition

$$H = Y \oplus Z$$

so that

- We have, for $y \in Y$,

  $$((A - \lambda L)y, y) \leq -n(\lambda)\|y\|^2$$

  and, for $z \in Z$,
  $$((A - \lambda L)z, z) \geq m(\lambda)\|y\|^2$$

  with $n(\lambda) > 0$ and $m(\lambda) > 0$.

Remark that we can take $Y = \{0\}$ if $\lambda < \lambda_1$.

Behavior on Z

Thus I_λ, when restricted to Z, is positive in a vicinity of 0, this being so uniformly:

Proposition

We have, for $\lambda \in]-\infty, 0]$ kept fixed,that there exists $\rho_\lambda > 0$ and $\alpha_\lambda > 0$ such that

$$I_\lambda(z) \geq \alpha_\lambda > 0, \qquad \forall z \in Z \text{ with } \|u\| = \rho_\lambda.$$

Palais-Smale condition

Proposition

For $\lambda \in]-\infty, 0] \setminus \sigma(-\Delta + v)$ kept fixed, the energy I_λ satisfies the Palais-Smale condition.

- For $\lambda < \lambda_1$, this is well-known.
- For λ inside the gaps, this is not quite elementary.
- We could not prove the Palais-Smale condition when $\lambda < 0$ is an eigenvalue.

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The circle $\mathscr{B}$ and the set $\mathscr{A}_m$

- $\mathscr{B} := \{z \in H \mid \|z\| = \rho_\lambda\}$;
- F_m a m-dimensional subspace of H, $m = 1, 2, 3, \ldots$. Then

Proposition

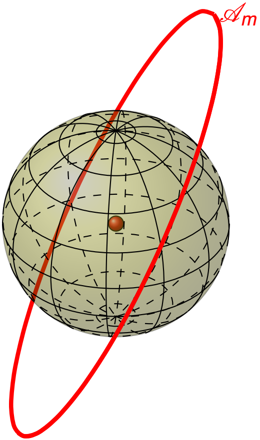
$\exists R > \rho_\lambda$ with

$$I_\lambda(u) \leq 0, \qquad \forall u \in F_m \text{ with } \|u\| = R.$$

- $\mathscr{A}_m := \{u \in H \mid 0 < \|u\| < R\}$ and

$$\partial\mathscr{A}_m := \{0\} \cup \{u \in H \mid \|u\| = R\}.$$

Odd linking



Proposition

$\gamma(\mathscr{A}_m) \cap \mathscr{B}$ is of genus $\geq m$, $\forall \gamma \in \Gamma_m$, where Γ_m is the set of all

- *(odd) homeomorphisms $\gamma : H \to H$*
- *with $\gamma(u) = u$, $\forall u \in \partial\mathscr{A}_m$.*

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Critical levels

By the deformation lemma, we get

Proposition

For $m = 1, 2, 3, \ldots$,

$$d_{m,0}(\lambda) := \inf_{\gamma \in \Gamma_m} \max_{u \in \mathscr{A}_m} I_\lambda(\gamma(u))$$

is a critical level of the energy I_λ with

$$0 < \alpha_\lambda \leq d_{1,0}(\lambda) \leq d_{2,0}(\lambda) \leq d_{3,0}(\lambda) \leq \cdots .$$

Critical levels, but missing multiplicity

Remark

Since $d_{1,0}(\lambda) = d_{2,0}(\lambda) = d_{3,0}(\lambda) = \cdots$ is possible, we can say nothing about multiplicity

First way-out

Show that $\lim_{m\to\infty} d_{m,0}(\lambda) = \infty$.
This approach is not suitable for multiple bifurcation since such an analysis is based on

$$\lim_{\lambda \nearrow \lambda_1} d_{m,0}(\lambda) = 0$$

for some m. This is not important for the first eigenvalue, since this one is simple, but it will be important later on.

Critical levels, but still missing multiplicity

Remark

Since $d_{1,0}(\lambda) = d_{2,0}(\lambda) = d_{3,0}(\lambda) = \cdots$ is possible, we can say nothing about multiplicity

Better way-out

Use the approach of Amborsetti-Rabinowitz
This approach will no longer be possible when $\dim Y = \infty$.

My way-out

Put

- $G_0 := \{\varnothing\}$
- $G_j = \{U \in H \mid U = -U, U \text{ open}, 0 \notin \bar{U}, U \text{of genus } \leq j\}$, j=1,2,3,. . . , m-1.

By the deformation lemma again,

Proposition

For $j = 0, 1, \ldots, m-1$ the values

$$d_{m,j}(\lambda) := \inf_{\gamma \in \Gamma_m, U \in G_j} \max_{u \in \mathscr{A}_m \setminus U} I_\lambda(\gamma(u))$$

is a critical value of the energy I_λ with

$$0 < \alpha_\lambda \leq d_{m,m-1}(\lambda) \leq d_{m,m-2}(\lambda) \leq \cdots \leq \underbrace{d_{m,0}(\lambda)}_{=c_m(\lambda)}.$$

My way-out

Proposition

If any of these levels coincide, then there is an infinite number of critical points of I_λ that correspond to this specific level.

Remark

In fact, we have the usual multiplicity:
If 3 of these value coincide, the corresponding set of critical points is of genus ≥ 3, a.s.o.

My way-out

- This approach is well suited for the analysis of multiple bifurcations (see later);
- This approach can be used even when $\dim Y = \infty$.

Bifurcation at λ_1

- Choose $F_1 := \mathrm{span}\{w_1\}$, where w_1 is an eigenfunction corresponding to the eigenvalue λ_1.
- Compute, for $\lambda < \lambda_1$,
$$\max_{u \in \mathscr{A}_1} I_\lambda(u)$$
and get $d_{1,0}(\lambda) \leq \mathrm{const} \cdot (\lambda - \lambda_1)^{1+\sigma/2}$.
- Using the monotone dependence of $d_{1,0}(\lambda)$ on λ to get L^2-bifurcation:
$$\lim_{\lambda \nearrow \lambda_1} |u_{1,\lambda}|_{L^2} = 0.$$

Outline

The circle $\mathscr{B}$ and the set $\mathscr{A}_m$

- $\mathscr{B} := \{z \in Z \mid \|z\| = \rho_\lambda\}$;
- E_m a m-dimensional subspace of H, $m = 1, 2, 3, \ldots$ and put $F_m := Y \oplus E_m$. Then

Proposition

$\exists R > \rho_\lambda$ *with*

$$I_\lambda(u) \leq 0, \qquad \forall u \in F_m \text{ with } \|u\| = R.$$

- $\mathscr{A}_m := \{u \in F_m \mid 0 < \|u\| < R\}$ and

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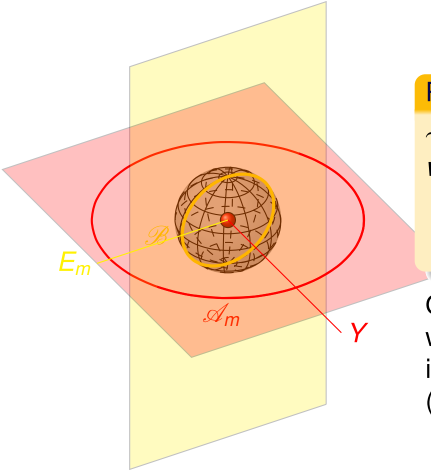

Proposition

$\gamma(\mathscr{A}_m) \cap \mathscr{B}$ is of genus $\geq m$, $\forall \gamma \in \Gamma_m$, where Γ_m is the set of all

- *odd homeomorphisms $\gamma : H \to H$*
- *with $\gamma(u) = u$, $\forall u \in \partial \mathscr{A}_m$.*

On can replace odd by $\gamma(u) = u$ whenever $I_\lambda(u) \leq 0$ if one is interested in the existence of only one solution (at least).

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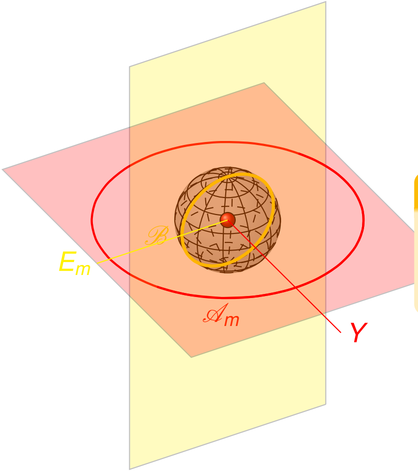


Proposition

$\forall \gamma \in \Gamma_m,$

$\gamma(\mathscr{A}_m) \cap \mathscr{B}$ *is of genus* $\geq \dim E_m = m$

We can now proceed as for $\lambda < \lambda_1$!!!

Critical levels

By the deformation lemma, we get

Proposition

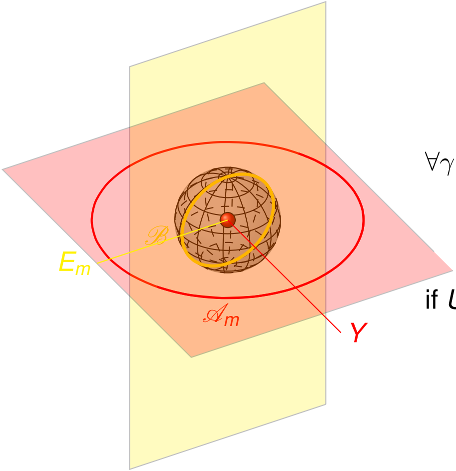
For $m = 1, 2, 3, \ldots$,

$$d_{m,0}(\lambda) := \inf_{\gamma \in \Gamma_m} \max_{u \in \mathscr{A}_m} I_\lambda(\gamma(u))$$

is a critical level of the energy I_λ *with*

$$0 < \alpha_\lambda \leq d_{1,0}(\lambda) \leq d_{2,0}(\lambda) \leq d_{3,0}(\lambda) \leq \cdots .$$

In order to get multiplicity



$\forall \gamma \in \Gamma_m$

$$\sup_{u \in \mathscr{A}_m \setminus U} I_\lambda(\gamma(u)) \geq \alpha_\lambda > 0,$$

if $U \in G_j$ $(j = 0, 1, 2, \ldots, m-1)$.

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$$0 < \alpha_\lambda \leq d_{m,m-1}(\lambda) \leq d_{m,m-2}(\lambda) \leq \cdots \leq \underbrace{d_{m,0}(\lambda)}_{=c_m(\lambda)}.$$

Multipicity result

Proposition

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Remark

In fact, we have the usual multiplicity:
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Important remarks

- This approach is well suited for the analysis of multiple bifurcations (see later);
- This approach can be used even when $\dim Y = \infty$.

Bifurcation at λ_{i+1}

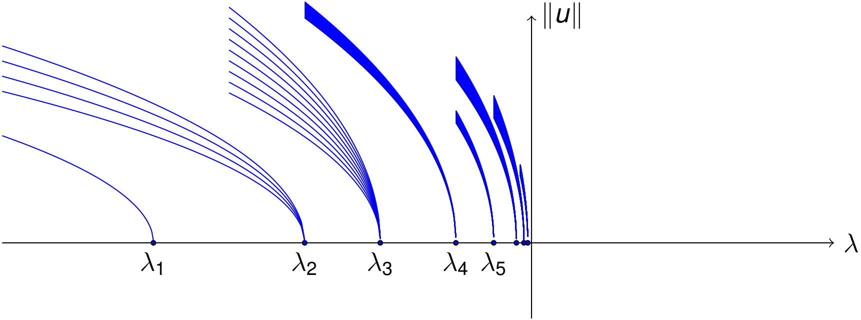
- Choose $F_m := \text{span}\{w_1, \dots, w_m\}$, where $w_1, \dots w_m$ are eigenfunctions corresponding to the eigenvalue λ_{i+1}.
- Compute, for $\lambda \in]\lambda_i, \lambda_{i+1}[$,

$$\max_{u \in \mathscr{A}_m} I_\lambda(u)$$

  and get $d_{m,0}(\lambda) \leq \text{const} \cdot (\lambda - \lambda_{i+1})^{1+\sigma/2}$.
- Using the monotone dependence of $d_{m,0}(\lambda)$ on λ to get L^2-bifurcation of multiplicity m:

$$\lim_{\lambda \nearrow \lambda_{i+1}} |u_{j,\lambda}|_{L^2} = 0, \qquad j = 1, 2, \dots, m.$$

bifurcation diagram



If $\lim_{n\to\infty} \lambda_n = 0$, 0 is a bifurcation point, too!