Separable solutions of p-Laplace type equations in cones and quasilinear problems on the sphere

Alessio Porretta
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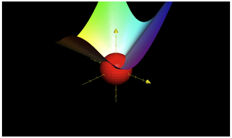

A. Porretta & L. Véron:*Separable p-harmonic functions in a cone and related quasilinear equations on manifolds*, J. Eur. Math. Soc. '09
A. Porretta & L. Véron: *Separable solutions of quasilinear Lane-Emden equations*, preprint.

Motivation and setting of the problem

Let C_S be a cone in $\mathbb{R}^N$ with vertex 0 and opening $S \subset S^{N-1}$, where S is a smooth subdomain on the sphere.

Pb: Construct positive solutions in C_S (vanishing on the lateral boundary) in the form of separable variables

$$u(x) = r^{-\alpha}\omega(\sigma)$$

for the p–harmonic equation

$$u \geq 0\,, \qquad -\Delta_p u := -\mathrm{div}\left(|Du|^{p-2}Du\right) = 0 \qquad \text{in } C_S \setminus \{0\}$$

or the quasilinear Lane-Emden equation

$$u \geq 0\,, \qquad -\Delta_p u = u^q\,, \quad q > p-1\,.$$

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Motivation: study of isolated boundary singularities of solutions of

$$\begin{aligned} -\Delta_p v = f(x,v) \qquad & \text{in } \Omega, \\ v = 0 \qquad & \text{on } \partial\Omega \setminus \{x_0\}. \end{aligned}$$

The p-harmonic case.

Theorem (P. Tolksdorf '83)

There exists a unique $\alpha := \alpha_S > 0$ and a unique (up to an homothethy) positive $\omega \in C^1(\bar{S}) \cap C^2(S)$ such that $u = r^{-\alpha}\omega(\sigma)$ is p-harmonic in C_S (and zero on the lateral boundary).

- Similarly, there exists a unique $\tilde{\alpha}_S < 0$ such that $u = r^{-\alpha}\omega(\sigma)$ is p-harmonic (the regular solution).
- The value of α_S appears in Liouville type problems in cones ([Berestycki-Capuzzo Dolcetta-Nirenberg], [Fraas-Pinchover]).
- Unfortunately, the explicit value of α_S is rarely known. (Ex: $p = 2$, $S = S_+$ half sphere, then $\alpha_S = N - 1$) However, the role of α_S is important as that of an eigenvalue.

- The value of α_S also plays a crucial role for the Lane-Emden equation (see [Bidaut Verón-Jazar-Véron]):

A necessary condition for $\exists$ of sol. $u = r^{-\alpha}\omega(\sigma)$ of

$$-\Delta_p u = u^q \qquad \text{in the cone } C_S$$

is that $\qquad \alpha = \frac{p}{q-(p-1)} < \alpha_S$

Note that this is a condition relating q and S (opening of the cone): $q-(p-1) > \frac{p}{\alpha_S}$

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$$\text{is that} \qquad \alpha = \frac{p}{q-(p-1)} < \alpha_S$$

Note that this is a condition relating q and S (opening of the cone): $q - (p-1) > \frac{p}{\alpha_S}$

(the condition is also sufficient in dimension $N = 2$)

(when $p = 2$ and $S = S_+$ half sphere, optimal sufficient conditions are given in [Bidaut Verón-Ponce-Véron])

One can check: $u(x) = r^{-\alpha}\omega(\sigma)$ is p-harmonic in the cone C_S (and zero on the lateral boundary) if and only if (α, ω) satisfy

$$\begin{cases} -div\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ \qquad\qquad = \alpha\left(\alpha(p-1) + p - N\right)(\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases} \tag{1}$$

where ∇ and *div* are covariant derivative and divergence operator on S^{N-1}.

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Tolksdorf' s result $\Rightarrow \exists\,!$ of (α, ω) sol. of a quasilinear pb. on the sphere

Depsite this problem is intrinsic on the sphere, the approach of P. Tolksdorf uses self-similarity arguments and properties of solutions of the (euclidean) p-Laplace equation.

In Tolksdorf's proof, the existence of (α, ω) is deduced by constructing a self-similar sol. in the unit cone ($u(R\,x) = R^\alpha\, u(x)$) and defining $\omega(\sigma) := \frac{u(R\sigma)}{R^\alpha}$. Uniqueness of α, ω is proved next using Harnack inequalities in the infinite cone (Praghmen-Lindelhof principle).

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Pb: Is there an intrinsic construction of (α, ω) ? Does this problem have an independent meaning on S^{N-1}?

Note that Note that problem

$$\begin{cases} -div\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ \qquad\qquad = \alpha\,(\alpha(p-1) + p - N)\,(\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases}$$

is a kind of "nonlinear eigenvalue problem"

(invariant by dilations of ω) - but it is not variational (except if $p = 2$) !

When $p = 2$, the equation

$$-div\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ = \alpha(\alpha(p-1) + p - N)(\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega$$

is exactly an eigenvalue problem

$$-\Delta_g\omega = \alpha(\alpha + 2 - N)\omega \qquad \text{in } S \subset S^{N-1} \tag{2}$$

where Δ_g is the Laplace-Beltrami operator.

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$$\alpha(\alpha + 2 - N) = \lambda_{1,S}$$

when $\lambda_{1,S}$ is the first eigenvalue on S.

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when $\lambda_{1,S}$ is the first eigenvalue on S.

Note in the case $p = 2$:

- ω is precisely an eigenfunction
- α is not precisely an eigenvalue, but is obtained in terms of λ_1

(α solves an equation $F(\alpha, \lambda_1) = 0$)

$\exists$ of sol. $u(x) = r^{-\alpha}\omega(\sigma) \longrightarrow$ eigenvalue-type problems in S^{N-1}.

What if $p \neq 2$? Key point: set

$$v = -\frac{1}{\alpha} \ln \omega$$

Then the equation

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is transformed into

$$-div \left(\left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} \nabla v \right) + \alpha(p-1) \left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} |\nabla v|^2 \\ = -\left(\alpha(p-1) + p - N\right) \left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} \quad \text{in } S$$

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Divide by $\left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}}$ .....

We see that $v = -\frac{1}{\alpha} \ln \omega$ solves

$$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \alpha(p-1)|\nabla v|^2 = -\left(\alpha(p-1) + p - N\right)$$

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- In the equation of v, the case $p = 2$ and $p \neq 2$ are very similar
- The number $(\alpha(p-1) + p - N)$ has a role of "ergodic constant": *given any $\alpha > 0$, is there some (unique?) λ_α:*

$$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \alpha(p-1)|\nabla v|^2 = -\lambda_\alpha$$

has a solution v ?
Important: with the boundary condition $v \to +\infty$ on ∂S !

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  has a solution v ?
  Important: with the boundary condition $v \to +\infty$ on ∂S !
- Recall that $\omega = e^{-\alpha v}$ and $u = r^{-\alpha}\omega$ is p-harmonic iff $\lambda_\alpha = (\alpha(p-1)+p-N)$.

The heart of our construction is the following

Theorem (P-V)

Let $S \subset S^{N-1}$ be a smooth bounded open subdomain. Then for any $\alpha > 0$ there exists a unique $\lambda_\alpha > 0$ such that the problem

$$\begin{cases} -\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \alpha(p-1)|\nabla v|^2 = -\lambda_\alpha \\ v(\sigma) \to +\infty \qquad \text{as } \sigma \to \partial S \end{cases}$$

admits a solution $v \in C^2(S)$, and v is unique up to an additive constant.

Furthermore, the map $\alpha \mapsto \lambda_\alpha$ is continuous, decreasing and $\lambda_\alpha \to \infty$ as $\alpha \to 0^+$.

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This result has an intrinsic independent interest:

- Our proof applies replacing S^{N-1} with a general $N-1$-dimensional Riemannian manifold (M, g).

• This result extends [J.M.Lasry-P.L.Lions '89] (where $p = 2$ and $S \subset \mathbb{R}^N$). It is new when $p \neq 2$ even in the euclidean case.

When $p = 2$, the problem

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This is a classical connection (through logarithmic tranform) between the first eigenvalue and the ergodic constant of stochastic control problems

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \overset{v=-\ln u}{\longleftrightarrow} \quad \begin{cases} -\Delta v + |\nabla v|^2 = -\lambda_1 & \text{in } \Omega \\ v \to +\infty & \text{on } \partial\Omega \end{cases}$$

So-called stochastic control interpretation of the first eigenvalue [C.J. Holland '77, J.M. Lasry-P.L.Lions '89]

(see also Donsker-Varadhan, W.H. Fleming-McEneaney '95, W. H. Fleming-S.J. Sheu '97, ...)

As a Corollary, we deduce Tolksdorf's result. Recall

$$v = -\frac{1}{\alpha}\ln\omega \quad \leftrightarrow \quad \omega = e^{-\alpha\, v}$$

We proved that, for any given $\alpha > 0$, there exists a unique $\lambda_\alpha > 0$:

$$\begin{cases} -div\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \alpha\,\lambda_\alpha(\alpha^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases}$$

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Somehow, for any α the role of "eigenvalue" is played by λ_α.

Tolksdorf's problem becomes:

$u(x) = r^{-\alpha}\omega(\sigma)$ is p-harmonic in the cone

if and only if $\lambda_\alpha = \alpha(p-1) + p - N$

But $\alpha \mapsto \lambda_\alpha$ is continuous, decreasing and unbounded....

Therefore, the mapping

$$\varphi(\alpha) := \lambda_\alpha - \alpha(p-1)$$

is continuous, decreasing and such that $\varphi(0) = +\infty$, $\varphi(+\infty) = -\infty$.

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By continuity, the equation

$$\lambda_\alpha - \alpha(p-1) = Y$$

has a unique sol. for every Y.

When $Y = p - N$ we get the unique $\alpha = \alpha_S > 0$ which makes $u = r^{-\alpha}\omega$ p-harmonic in the cone.

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Rmk: The monotonicity of the map $\alpha \mapsto \lambda_\alpha$ gives a typical monotonicity property of eigenvalues:

$$\text{if } S, S' \subset S^{N-1}, \qquad S \subset S' \Rightarrow \alpha_S \geq \alpha_{S'}$$

Corollary (P-V)

There exists a unique $\alpha > 0$ such that

$$\lambda_\alpha = \alpha(p-1) + p - N \tag{3}$$

As a consequence, for any subdomain S there exists a unique $\alpha_S > 0$ and a unique (up to dilation) positive $\omega \in C^1(\overline{S}) \cap C^2(S)$: $u(x) = r^{-\alpha}\omega(\sigma)$ is p-harmonic in the cone C_S.

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Remarks:

- As in the case $p = 2$: ω is an eigenfunction, α is not exactly an eigenvalue but a solution of an equation $F(\alpha, \lambda_\alpha) = 0$ where λ_α is an eigenvalue.

- When $p = 2$ we have $\lambda_\alpha = \frac{\lambda_1}{\alpha}$ and (3) is the algebraic equation $\alpha(\alpha + 2 - N) = \lambda_{1,S}$.

Ideas of the proof

The proof of this Theorem stands on the following steps:

- As is typical for ergodic-type problems, we start from

$$\begin{cases} \varepsilon\, v_\varepsilon - \Delta_g v_\varepsilon - (p-2)\frac{D^2 v_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon}{1+|\nabla v_\varepsilon|^2} + \alpha(p-1)|\nabla v_\varepsilon|^2 = 0 \\ v_\varepsilon(\sigma) \to +\infty \qquad \text{as } \sigma \to \partial S \end{cases}$$

and then we let $\varepsilon \to 0$.

What happens in such models is that
- v_ε has a complete blow-up as $\varepsilon \to 0$

On the other hand,

-$\varepsilon\, v_\varepsilon$ remains bounded (locally) by max. principle

-$|\nabla v_\varepsilon|$ remains locally bounded due to the barrier effect of the absorption term.

Therefore we have

$$\varepsilon \nabla v_\varepsilon \to 0 \quad \text{locally uniformly,}$$

hence, up to subsequences,

$$\varepsilon\, v_\varepsilon \text{ converges to a constant } \lambda_\alpha$$

If we fix $\sigma_0 \in S$, then, locally uniformly,

$$v_\varepsilon(\cdot) - v_\varepsilon(\sigma_0) \quad \text{converges to a function } v$$

and v solves

$$\lambda_\alpha - \Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \alpha(p-1)|\nabla v|^2 = 0$$

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with the boundary behaviour $v \to +\infty$ on ∂S

Key technical points:

- compactness relies on interior gradient estimates:

  For every compact subset $S' \subset\subset S$, we have

$$\|\nabla v_\varepsilon\|_{L^\infty(S')} \leq \frac{K}{\mathrm{dist}(S', S)}$$

  To get the gradient bound, we use the (intrinsic) Weitzenböck formula

$$\frac{1}{2}\Delta_g |\nabla v|^2 = \|D^2 v\|^2 + \nabla(\Delta_g v) \cdot \nabla v + Ricc_g(\nabla v, \nabla v)$$

  and the classical Bernstein's method

  (max. principle applied to $|\nabla v|^2$)

- Uniqueness of (λ_α, v)

[Rmk: Uniqueness of (λ_α, v) implies that the convergence holds for the whole sequence v_ε]

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Two main ingredients:

(i) the strong maximum principle

Rmk: $A(v) := -\Delta_g v - (p-2)\dfrac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2}$ is nondegenerate

$$A(v_1) - A(v_2) + \alpha\left[|\nabla v_1|^2 - |\nabla v_2|^2\right] = -(\lambda_\alpha^1 - \lambda_\alpha^2)$$
$$\lambda_\alpha^1 \neq \lambda_\alpha^2 \quad \Rightarrow \quad v_1 - v_2 \equiv \text{const.}$$

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$$\lambda_\alpha^1 \neq \lambda_\alpha^2 \quad \Rightarrow \quad v_1 - v_2 \equiv \text{const.}$$

$$\Rightarrow \begin{cases} \text{uniqueness of } \lambda_\alpha \\ \text{uniqueness (up to an additive constant)} \\ \quad \text{of the boundary blow-up solution } v. \end{cases}$$

(ii) Detailed estimates on the boundary blow-up of v, ∇v in order to handle the difference of solutions near the boundary.

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In particular, we need precise gradient estimates:

$$\frac{\gamma_1}{\mathrm{dist}(\sigma, \partial\Sigma)} \leq |\nabla v(\sigma)| \leq \frac{\gamma_2}{\mathrm{dist}(\sigma, \partial\Sigma)}$$

which we prove using $C^{1,\alpha}$ estimates up to the boundary for p-Laplace type equations.

Properties of the mapping $\alpha \mapsto \lambda_\alpha$ follow from the construction of the couple (λ_α, v) sol. of the ergodic problem

$$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \alpha(p-1)|\nabla v|^2 = -\lambda_\alpha$$

one checks that

- $\alpha \mapsto \lambda_\alpha$ is decreasing (since $\lambda_\alpha = \lim_{\varepsilon \to 0} \varepsilon\, v_\varepsilon$)
- $\alpha \mapsto \lambda_\alpha$ is continuous (stability of the ergodic constant constant is consequence of its uniqueness)
- we have $\lambda_\alpha \to +\infty$ when $\alpha \to 0$.

Comments

- Our proof of Tolksdorf's result is not easier. However we provide an intrinsic interpretation of the unique couple (α_S, ω_S) such that $u(r, \sigma) = r^{-\alpha}\omega(\sigma)$ is p-harmonic in the cone C_S, and a new construction of (α, ω) (valid in general manifolds).
- The *log*-transform reminds of the useful connection between the first eigenvalue and the ergodic constant of stochastic control problems

  Our approach suggests that in some cases it can be useful *to embed eigenvalue problems into the larger family of ergodic problems*
- Our construction can be useful to understand the role of α_S in the Lane-Emden problem.

The Lane-Emden equation

$$-\Delta_p u = u^q\,, \quad \text{in the cone } C_S, \text{ with } q > p-1.$$

A positive (singular) solution $u = r^{-\alpha}\omega(\sigma)$ exists iff (α, ω) satisfy the quasilinear pb. on the sphere:

$$-div_g\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) =$$

$$= \alpha(\alpha(p-1)+p-N)(\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

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$$= \alpha(\alpha(p-1)+p-N)(\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

Recall the necessary conditions: $\alpha = \frac{p}{q-(p-1)}$ and $\alpha < \alpha_S$

The Lane-Emden equation

$$-\Delta_p u = u^q \,, \quad \text{in the cone } C_S, \text{ with } q > p-1.$$

A positive (singular) solution $u = r^{-\alpha}\,\omega(\sigma)$ exists iff (α, ω) satisfy the quasilinear pb. on the sphere:

$$-div_g\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) =$$

$$= \alpha(\alpha(p-1)+p-N)(\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

Recall the necessary conditions: $\alpha = \frac{p}{q-(p-1)}$ and $\alpha < \alpha_S$

But our construction of α_S implies:

$$\alpha < \alpha_S \quad \Longleftrightarrow \quad (\alpha(p-1)+p-N) < \lambda_\alpha$$

where λ_α is the unique "eigenvalue":

$$-div_g\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) = \alpha\lambda_\alpha(\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega$$

Observe the analogy with the euclidean case:

$$\exists \text{ pos. sol. of } \quad -\Delta_p u = \lambda\, u^{p-1} + u^q \quad \Rightarrow \quad \lambda < \lambda_1(-\Delta_p, \Omega)$$

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Theorem

Assume that $\alpha = \frac{p}{q-(p-1)} < \alpha_S$.

(i) If $q < \frac{(N-1)p}{N-1-p} - 1$ *(critical exponent in dim.* $N-1$*), then* $\exists$ *a sol. of*

$$-div_g\left((\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) =$$
$$= \alpha(\alpha(p-1)+p-N)(\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

(hence $\exists$ *a separable sol. of* $-\Delta_p u = u^q$ *in the cone* C_S*).*

(ii) If S is "star shaped with respect to the North pole", then there is no solution when $q = \frac{(N-1)p}{N-1-p} - 1$.

Ideas of the proof

- For the nonexistence part, we use a Pohozaev type identity on the sphere.
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  Rmk: we only conclude nonexistence for the critical value $q = \frac{(N-1)p}{N-1-p} - 1$ (if $p = 2$ nonexistence holds for any q supercritical)

  Pohozaev identity takes the form:

$$\int_{\partial S} |\omega_\nu|^p \phi_\nu dS = A \int_S \omega^{q+1} \phi \, d\sigma$$

$$+B \int_S \gamma_\omega |\nabla' \omega|^2 \phi \, d\sigma + C \int_S \gamma_\omega \omega^2 \phi \, d\sigma,$$

  where $\gamma_\omega = (\alpha^2 \omega^2 + |\nabla' \omega|^2)^{p-2/2}$ and A, B, C depend on α, q, p, N.

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  where $\gamma_\omega = (\alpha^2 \omega^2 + |\nabla' \omega|^2)^{p-2/2}$ and A, B, C depend on α, q, p, N. Strange miracle:

$$q \text{ critical} \iff A = B = C = 0$$

- For the existence part, we use topological degree (as in [DeFiguereido-Lions-Nussbaum], [Quaas-Sirakov]) and a priori estimates for p-Laplace Lane-Emden equations ([Serrin-Zou], [Zou], with similar method as [Gidas-Spruck]).

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  In the topological degree argument, the role of λ_α as eigenvalue is important. Recall: we look for solutions of

$$-div_g \left((\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega \right) = \\ = \alpha \underbrace{\left(\alpha(p-1) + p - N\right)}_{<\lambda_\alpha} (\alpha^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

Roughly speaking, the degree homotopy takes the form

$$-div_g\left((\alpha^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega\right)=$$
$$=\alpha(c_\alpha+t)(\alpha^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega+(\omega+t)^q$$

where $c_\alpha=\alpha(p-1)+p-N$. Recall that we are in the range

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- The "eigenvalue" meaning of λ_α implies
no solution for t small and ω small $\Rightarrow$ index=1 on B_r for small r.

-A priori estimates +no solution for t large $\Rightarrow$ index=0 on B_R for large R.

Hence, there exists a solution on $B_R\setminus B_r$.