SEMILINEAR ELLIPTIC EQUATIONS WITH SINGULAR NONLINEARITIES

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We study existence and nonexistence of solutions for the following semilinear elliptic problem with a singular nonlinearity:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\gamma > 0$ is a real number, f is either a nonnegative function belonging to some Lebesgue space, or a nonnegative bounded Radon measure, and M is a bounded elliptic matrix; i.e., there exist $0 < \alpha \leq \beta$ such that

$$\alpha\,|\xi|^2 \leq M(x)\xi\cdot\xi\,, \qquad |M(x)| \leq \beta\,, \tag{0.2}$$

for every ξ in $\mathbb{R}^N$, for almost every x in Ω. A solution of (0.1) is a function u in $W^{1,1}_0(\Omega)$ such that

$$\forall \omega \subset\subset \Omega\ \exists c_\omega : u \geq c_\omega > 0 \text{ in } \omega, \tag{0.3}$$

and such that

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi = \int_\Omega \frac{f\,\varphi}{u^\gamma} \qquad \forall \varphi \in C^1_0(\Omega)\,. \tag{0.4}$$

Note that the right hand side is well defined by (0.3) since φ has compact support.

Problem (0.1) is strongly connected to the quasilinear singular problem (studied by D. Arcoya and co., L. Boccardo, P. Martinez)

$$\begin{cases} -\Delta u + A\,\dfrac{|\nabla u|^2}{u} = h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.5}$$

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with $0 < A < 1$, and h a nonnegative function. Indeed, if we define $v = u^{1-A}$, and formally perform a change of variable, then v is a solution of problem (0.1) with $f(x) = (1-A)\,h(x)$, $\gamma = \frac{A}{1-A}$, and M the identity matrix.

We will prove some existence and regularity results for problem (0.1), depending on γ (more precisely, the cases $\gamma = 1$, $\gamma > 1$ and $\gamma < 1$ will be studied separately, the first having some features in common with the second and the third), and on the summability of f. If f is a bounded Radon measure, we will prove nonexistence results; for example, we will prove that no solution exists if $f = \delta_{x_0}$, the Dirac mass concentrated at x_0 in Ω, for every $\gamma > 0$.

Let f be a nonnegative measurable function (not identically zero), let $n \in \mathbb{N}$, let $f_n(x) = \min(f(x), n)$ and consider the following problem:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \dfrac{f_n}{(u_n + \frac{1}{n})^{\gamma}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.6}$$

Lemma 0.1. *The sequence u_n is increasing with respect to n, $u_n > 0$ in Ω, and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent on n) such that*

$$u_n(x) \ge c_\omega > 0 \qquad \text{for every } x \text{ in } \omega, \text{ for every } n \text{ in } \mathbb{N}. \tag{0.7}$$

Remark 0.2. If u_n and v_n are two solutions of (0.6), repeating the argument of the first part of the proof of Lemma 0.1 shows that $u_n \le v_n$. By symmetry, this implies that the solution of (0.6) is unique.

If $\gamma < 1$, an *a priori* estimates on u_n in $H_0^1(\Omega)$ can be obtained only if f is more regular than $L^1(\Omega)$.

Theorem 0.3. *Let $\gamma < 1$ and let f be a nonnegative (not identically zero) function in $L^m(\Omega)$, with $m = \frac{2N}{N+2+\gamma(N-2)} = \left(\frac{2^*}{1-\gamma}\right)'$. Then there exists a solution u in $H_0^1(\Omega)$ of (0.1).*

Remark 0.4. In [BO-Houston] the authors studied the problem

$$-\mathrm{div}(M(x)\,\nabla u) = \rho(x)\,u^{\theta}\,,$$

with ρ a nonnegative function in $L^m(\Omega)$, and $0 < \theta < 1$, proving existence of solutions in $H_0^1(\Omega)$ if $m \ge \left(\frac{2^*}{1+\theta}\right)'$. The previous theorems allow us to extend that result to $-1 < \theta < 1$.

Remark 0.5. If the matrix $M(x)$ is symmetric, and if f belongs to $L^m(\Omega)$, with $m > \left(\frac{2^*}{1-\gamma}\right)'$, the solution of (0.1) given by Theorem 0.3 is the minimum of the functional

$$J(v) = \frac{1}{2}\int_\Omega M(x)\nabla v\cdot\nabla v - \frac{1}{1-\gamma}\int_\Omega f\,v^{1-\gamma}\,,\quad v\in H_0^1(\Omega)\,,$$

which is well defined since $\gamma<1$. Indeed, if we consider the functional

$$J_n(v) = \frac{1}{2}\int_\Omega M(x)\nabla v\cdot\nabla v - \frac{1}{1-\gamma}\int_\Omega f_n\left(v^+ + \frac{1}{n}\right)^{1-\gamma}\,,\quad v\in H_0^1(\Omega)\,,$$

with $f_n=\min(f(x),n)$, then there exists a minimum u_n of J_n. From the inequality $J_n(u_n)\le J_n(u_n^+)$ one can prove that $u_n\ge 0$, so that u_n is a solution of the Euler equation for J_n, i.e., of (0.6). Therefore, by Lemma 0.1 and Remark 0.2, u_n is unique and increasing in n, satisfies (0.7) and, from the inequality $J(u_n)\le J_n(0)\le C$, it is bounded in $H_0^1(\Omega)$ (with the same proof of Theorem **??**). If u is the limit of u_n, letting n tend to infinity in the inequalities $J_n(u_n)\le J_n(v)$, one finds that $J(u)\le J(v)$, so that u is a minimum of J, and u is a solution of (0.1) (by Theorem 0.3). Since u satisfies (0.7), equation (0.1) can be seen as the Euler equation for J; note that J is not differentiable on $H_0^1(\Omega)$.

If $m<\left(\frac{2^*}{1-\gamma}\right)'$, we no longer have solutions in $H_0^1(\Omega)$, but in a larger Sobolev space (which depends on m).

Theorem 0.6. *Let $\gamma<1$, and let f belong to $L^m(\Omega)$, $1\le m<\frac{2N}{N+2+\gamma(N-2)}$. Then there exists a solution u of (0.1), with u in $W_0^{1,q}(\Omega)$, $q=\frac{Nm(\gamma+1)}{N-m(1-\gamma)}$.*

Theorem 0.7. *Let $\gamma=1$ and let f be a nonnegative function in $L^1(\Omega)$ (not identically zero). Then there exists a solution u in $H_0^1(\Omega)$ of (0.1), in the sense that*

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi = \int_\Omega \frac{f\,\varphi}{u}\qquad \forall\varphi\in C_0^1(\Omega)\,. \tag{0.8}$$

Theorem 0.8. *Let μ be a nonnegative Radon measure concentrated on a Borel set E of zero harmonic capacity, and let g_n be a sequence of nonnegative $L^\infty(\Omega)$ functions that converges to μ in the narrow topology of measures. Let $\gamma=1$, and let u_n be the solution of (0.6) with g_n as datum. Then u_n converges weakly to zero in $H_0^1(\Omega)$.*

Theorem 0.9. *Let $\gamma>1$ and let f be a nonnegative function in $L^1(\Omega)$ (not identically zero). Then there exists a solution u in $H^1_{\rm loc}(\Omega)$ of (0.1) (in the sense of (0.4)). Furthermore, $u^{\frac{\gamma+1}{2}}$ belongs to $H_0^1(\Omega)$ (this is the meaning of $u=0$ on the boundary of Ω).*

Remark 0.10. The case $\gamma > 1$ corresponds to $\frac{1}{2} < A < 1$ in problem (0.5).