Gradient bounds for elliptic problems singular at the boundary

Tommaso Leonori

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Let us consider the following class of second order Hamilton Jacobi equations:

$$-\alpha\,\Delta u + u + H(x, \nabla u) = 0 \quad \text{in } \Omega\,,$$

where Ω is a smooth (say C^2) bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$ and $H(x, p)$ is a Caratheodory function.

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We are interested in considering nonlinear Hamiltonians that are singular at the boundary.

Our aim is to prove gradient bounds for such class of equations.

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Rmk.: No boundary conditions are prescribed!

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- find suitable conditions in order to have uniqueness of solutions;
- study the behavior of solutions at the boundary;
- look at the vanishing viscosity (i.e. as $\alpha \to 0$);
- apply such estimates to a problem of large solutions in order to find secondary effects in the asymptotic expansion of the gradient.

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"*We want to constrain a Brownian motion in a given domain Ω by controlling its drift*".

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Typical examples of controls are constructed as functions of the distance to the boundary, that are singular at the boundary itself, i.e.

$$a(x) \sim \psi(d(x)) \quad \text{with} \quad \lim_{d(x)\to 0} |\psi(x)| = +\infty\,.$$

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This field has a privileged direction which reminds of the control mechanism acting basically in the normal direction.

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$$\Delta v = v + H(x, \nabla v) - f$$

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Theorem (T.L., A. Porretta - ARMA 2011)

Let $c(x) \in W^{1,\infty}(\Omega)$, $B(x) \in W^{1,\infty}(\Omega)^N$ *with*

$$B(x) \cdot \nu \geq \sigma > 0, \quad B(x) \cdot \tau = 0 \qquad \text{at } \partial\Omega$$

and $\sigma > \alpha$ *and assume that* $f(x) \in W^{1,\infty}_{loc}(\Omega)$ *satisfies near the boundary*

$$|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \qquad \text{where } \int_0^1 \frac{\rho(s)}{s}\, ds < \infty .$$

Then there exists a solution u of (E_α) *in* $u \in C^2(\Omega) \cap W^{1,\infty}(\Omega)$.

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For $\alpha = \sigma$ the same result holds true under stronger hypothesis on ρ, namely

$$\int_0^1 \frac{1}{s} \left(\int_0^s \frac{\rho(\tau)}{\tau} d\tau \right) ds < \infty .$$

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where $\Omega_n = \{x \in \Omega \,:\, d(x) > \frac{1}{n}\}$.

Idea of the Proof.

First, we approximate our equation in order to "desingularize" it, with solutions that satisfy a Neumann boundary condition at the interior of Ω, namely:

$$\begin{cases} -\alpha\Delta u_n + u_n + \frac{B(x)\cdot\nabla u_n}{d(x)} + c(x)|\nabla u_n|^2 = f(x) & \text{in } \Omega_n\,, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial\Omega_n\,, \end{cases}$$

where $\Omega_n = \{x \in \Omega \,:\, d(x) > \frac{1}{n}\}$.

We focus our attention on the function

$$w_n = |\nabla u_n|^2 e^{\theta(d)}$$

where θ is a bounded function (but its first derivative, in general, is singular at $d(x) = 0$).

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Thus in the direction of ∇u_n we have

$$\mu(x)\,\nu(x) \cdot \nabla u_n = D^2 u_n\,\nu(x) \cdot \nabla u_n + J(\nu(x))\,\nabla u_n \cdot \nabla u_n \qquad \text{on } \partial\Omega_n\,.$$

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$$\begin{aligned}\nabla w_n \cdot \nu = \nabla\Big(|\nabla u_n|^2 e^{\theta(d)}\Big)\cdot\nu \quad &= -\theta'(d) w_n + e^{\theta(d)}\,\nabla|\nabla u_n|^2\cdot\nu \\ &\leq -\theta'(d) w_n + 2\|D^2 d\|\,e^{\theta(d)}|\nabla u_n|^2\end{aligned}$$

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Thus (Hopf Lemma) the maximum of w_n is not achieved at the boundary of Ω_n.

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We now choose

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This implies

$$\sup_{\overline{\Omega}_n \setminus \Omega_\delta} |\nabla u_n|^2 \le \widetilde{C_0} + \sup_{\partial\Omega_\delta} |\nabla u_n|^2 .$$

For the case $c(x) \not\equiv 0$ we have to deal with

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Tedious computations yield to

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$$\forall K \subset\subset \Omega, \quad \sup_K |\nabla u_n|^2 \leq C\big(\text{dist}\,(K, \partial\Omega)\big) .$$

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Thus we deduce that

$$\exists c > 0 \;:\; |\nabla u_n|^2 \le c \quad \text{ in } \Omega .$$

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If $\exists\varphi \in C^2(\Omega)$ such that

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In the case of equation (E_α) it holds with $\varphi \sim \log(d)$.

Thus bounded solutions are unique!

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$$\begin{aligned} & -\alpha\Delta u_\varepsilon + u_\varepsilon + H(x, \nabla u_\varepsilon) \\ = & \,(1+\varepsilon)\,(-\alpha\Delta u + u) - \varepsilon\,(-\alpha\Delta\varphi + \varphi) + H(x, \nabla u_\varepsilon)\,. \end{aligned}$$

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Regularity and boundary conditions

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Since the solution belongs to $W^{1,\infty}(\Omega)$, there exists the trace at $\partial\Omega$ ad thus, for any $x_0 \in \partial\Omega$ we can rescale the equation near the boundary, we make a blow-up and it follows that the solution satisfies

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This in particular means that the homogeneous Neumann boundary condition is intrinsic in the equation.

Optimality of $\sigma \geq \alpha$: the Fichera condition

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Indeed for linear equations as

$$a_{ij}\partial^2_{ij} v + b_j v_j + cv = f \quad \text{in} \;\; \Omega$$

you can prescribe Dirichlet boundary data in the set

$$\Gamma_d = \left\{ x \in \partial\Omega \; : \; a_{ij}(x)\nu(x)\nu(x) > 0 \text{ or } \sum_j \left(b_j - \sum_i \partial_{x_i} a_{ij} \right) \nu_j > 0 \right\}$$

Assume that $c(x) \equiv 0$ in (E_α) and multiply the equation by $d(x)$, hence we have:

$$-\alpha d(x)\Delta u + d(x)u + B(x) \cdot \nabla u - d(x)f(x) = 0 \quad \text{in} \;\; \Omega\,.$$

Thus if $\sigma < \alpha$ our estimate should depend on the boundary value of u!

Generalizations

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- $-\alpha\,\Delta u + u + H(x, \nabla u) = 0 \quad \text{in } \Omega$, where $H(x, p)$ satisfies a local natural growth condition, and general assumptions, as

$$|H(x,p) - p \cdot H_p(x,p)| \leq C_0 |p|^2 + \frac{\rho(d)}{d}\,,$$

$$H_x(x,p) \cdot \frac{p}{|p|} \geq -\frac{\rho(d)}{d^2}|p| - \frac{\rho(d)}{d}|p|^2 - \frac{\rho(d)}{d^2}\,,$$

$$H_p(x,p) \cdot \nu(x) \geq \frac{\sigma}{d} - C_1 |p|\,,$$

and either

$$\sigma > \alpha\,, \qquad \text{and} \quad \int_0^1 \frac{\rho(t)}{t} dt < \infty\,,$$

or

$$\sigma = \alpha\,, \qquad \text{and} \quad \int_0^1 \frac{1}{t} \left(\int_0^t \frac{\rho(\tau)}{\tau} d\tau \right) dt < \infty\,.$$

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- Elliptic operator with smooth coefficients (say $W^{1,\infty}(\Omega)$)
- Weighted Lipschitz estimates (Hölder-type estimates, blow-up solutions...)

Stability (first order equation)

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In order to give a positive answer to such a question, we have to straight some hypotheses on the nonlinear term.

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- In order to get interior gradient bound $c(x)$ has to be positive in Ω (possibly vanishing at $\partial\Omega$);
- an approximation that involves a vanishing transport term i.e. the solutions of (E_0) are limit of

$$u - \alpha\Delta u + \alpha\frac{\nu\cdot\nabla u}{d(x)} + \frac{B(x)\cdot\nabla u}{d(x)} + c(x)|\nabla u|^2 = f(x) \quad \text{in } \Omega\,.$$

Theorem (T.L., A. Porretta - ARMA 2011)

Assume that $B(x) \in W^{1,\infty}(\Omega)^N$ *is such that* $B(x)\cdot\nu > 0$, *and* $f(x) \in W^{1,\infty}_{loc}(\Omega)$ *satisfies near the boundary*

$$|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \qquad \text{where} \int_0^1 \frac{\rho(s)}{s}\, ds < \infty\,.$$

Moreover suppose that $c(x) \in W^{1,\infty}_{loc}(\Omega)$ *is a positive function that satisfies the following condition near* $\partial\Omega$*:*

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Then there exists $u \in W^{1,\infty}(\Omega)$ *which is a viscosity solution of* (E_0) *and* $\frac{\partial u}{\partial \nu} = 0$ *(in the viscosity sense) at* $\partial\Omega$.

Application/motivation:

A stochastic control problem with state constraint.

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Let's go back to the model introduced by J.M. Lasry and P.L. Lions, and let us consider the SDE:

$$\begin{cases} dX_t = a_t dt + \sqrt{2} dB_t \\ X_0 = x \in \Omega\,. \end{cases}$$

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Among these controls, we want to select one that satisfies a criterion of optimality.

Thus, let $\mathcal{A}$ be the class of all (feedback) controls that keep the process X_t inside the domain Ω for any time $t > 0$ a.s..

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$$\begin{cases} -\Delta u + u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0\,. \end{cases}$$

Some results about large solutions

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- ([PV])

$$\begin{cases} \displaystyle\lim_{x \to x_0 \in \partial\Omega} \quad d(x)^{\frac{1}{q-1}} \nabla u(x) = (q-1)^{-\frac{1}{q-1}}\, \nu(x_0)\,, \\ \displaystyle\frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right) \end{cases}$$

Summary

The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary.

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where $\psi(s)$ is the solution of the ODE

$$\begin{cases} -\psi''(s) + |\psi'(s)|^q = 0 \quad s \in (0,1)\,, \quad 1 < q < 2\,, \\ \lim_{s \to 0^+} \psi(s) = +\infty\,. \end{cases}$$

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Theorem (L.-Porretta SIAM J.Math.Anal. 2008)

Let Ω be regular and let $H(\varsigma)$ be the mean curvature of $\partial\Omega$ computed at ς and $\overline{x} = Proj(x, \partial\Omega)$. Then $\forall\, 1 < q < 2$, as $d(x) \to 0$,

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and

$$\begin{cases} \frac{\partial u(x)}{\partial \tau} \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\ \frac{\partial u(x)}{\partial \tau} = O(|\log d|) & \text{if } q = \frac{3}{2}, \\ \frac{\partial u(x)}{\partial \tau} = O\left(d^{\frac{2q-3}{q-1}}\right) & \text{if } 1 < q < \frac{3}{2}. \end{cases}$$

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   (i.e. on the hypersurfaces parallel to $\partial\Omega$, it achives its maximum where the mean curvature is maximal).

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$$S = d^{-\frac{2-q}{q-1}}(x)\sum_{k=0}^{m}\sigma_k(x)d^k(x)\,, m > 0\,, \quad \sigma_0 = C^* = \frac{(q-1)^{-\frac{2-q}{q-1}}}{2-q}.$$

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Then we define $z = u - S$ and we look at the equation solved by z, i.e.

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$$\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}(x)} + \sum_{k=1}^{[\alpha]+1} \left[\frac{(k-\alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)$$

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In particular it is easy to see that

$$\sigma_1 = \frac{(q-1)^{-\frac{2-q}{q-1}}}{3-2q} \frac{\Delta d(x)}{2}$$

and recalling that $\Delta d(x)\Big|_{\partial\Omega} = (N-1)H(x)$ we deduce the result of the Theorem.

Key point: gradient bounds

Thus our aim is to prove a global Lipschitz estimate for the (unique) solution of

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Thus we are in the hypotheses of the previous Theorem. ■

Gracias!