Existence of Positive Solutions for some Nonlinear Systems

Giusi Vaira
vaira@sissa.it

SISSA

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Physical Motivations

In Quantum Mechanics, any state of a particle in 3-dimensional space can be described by a function

$$\psi(x,t) \in \mathbb{C}, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \rightsquigarrow \text{Wave Function}$$

$$|\psi|^2 \, dx$$

is the probability that the coordinates of the particle associated to ψ will find their values in the element dx.

$$\int_{\mathbb{R}^3} |\psi|^2 \, dx = 1 \qquad \text{Normalization Equation}$$

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The Schrödinger equation

$$\imath\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}$$

where $m > 0$, $\hbar$ is the Planck constant and $Q : \mathbb{R}^3 \to \mathbb{R}$ is the time independent potential energy of the particle at position $x \in \mathbb{R}^3$.

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Case of a Single Particle

$$\imath\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R} \qquad (\mathcal{SE})$$

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The Schrödinger equation

Case of Many Particles

$$\imath\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R} \qquad (\mathcal{NSE})$$

where $m > 0$, $\hbar$ is the Planck constant and $Q : \mathbb{R}^3 \to \mathbb{R}$ is the time independent potential energy of the particle at position $x \in \mathbb{R}^3$, $p > 1$.

The Schrödinger-Maxwell system

Let us assume now that ψ is a charged wave and we denote by $q > 0$ the electric charge.

Hence, the wave ψ interacts with its own electromagnetic field $\mathbf{E}, \mathbf{H}$.

Following the ideas introduced in

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Unknowns:

i) The wave function ψ;

ii) The gauge potentials

$$\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \qquad \phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$$

related to **E**, **H** by the Maxwell equations

$$\mathbf{E} := -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} := \nabla \times \mathbf{A}.$$

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Standing Waves interacting with a purely electrostatic field

We choose:

1. $$\psi(x,t) = u(x)e^{i\omega t}, \qquad u(x) \in \mathbb{R}, \omega > 0.$$

   that is called *standing wave*.
   Indeed, these solutions correspond to static situations in the sense that the density $|\psi(x,t)|^2 = u^2(x)$ does not change in time.
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Then we deal with the following system of equations:

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + V(x)u + q\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta\phi = qu^2 & x \in \mathbb{R}^3 \end{cases} \qquad (\mathcal{SP})$$

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where $K : \mathbb{R}^3 \to \mathbb{R}$ is a positive density charge and $V(x) = Q(x) + \hbar\omega$

Interaction with the gravitational field

If a particle of mass $m > 0$ moves in its own gravitational field

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If we look for standing waves $\psi(x,t) = u(x)e^{i\omega t}$ then we have to deal with the following system of equations:

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Schrödinger-Newton system

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Notations:

Here:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u,v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + uv] dx; \qquad \|u\|^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2\right] dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

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First of all we look for solution

$$(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$$

for the problem $(\mathcal{SP})$.

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$$\epsilon^2 := \frac{\hbar^2}{2m}.$$

- Existence Results for $\epsilon > 0$ small;
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It is well-known that, for all $u \in H^1(\mathbb{R}^3)$, the Poisson equation

$$-\Delta\phi = K(x)u^2$$

has a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ given by

$$\phi_u(x) = \frac{1}{|x|} * Ku^2 = \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u^2(y)\, dy.$$

Hence, inserting ϕ_u into the first equation of $(\mathcal{SP})$, we deal with the equivalent problem

$$-\epsilon^2 \Delta u + V(x)u + K(x)\phi_u u = |u|^{p-1}u, \qquad (\mathcal{SP}')$$

Remark

$u \in H^1(\mathbb{R}^3)$ is a solution of $(\mathcal{SP}') \Longrightarrow (u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SP})$

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The positive solutions $u_\epsilon \in H^1(\mathbb{R}^3)$ of $(\mathcal{SP}')$ founded for ϵ small are called Semiclassical States.
Interesting classes of semiclassical states are those which exihibit a concentration behavior around one or more special point.
These solutions are called Spikes.

Definition

A solution u_ϵ of $(\mathcal{SP}')$ concentrates at $x_0 \in \mathbb{R}^3$ (as $\epsilon \to 0$) provided

$$\forall\ \delta > 0, \quad \exists \epsilon_0 > 0,\ R > 0 : u_\epsilon(x) \leq \delta,\ \forall\ |x - x_0| \geq \epsilon R,\ \epsilon < \epsilon_0$$

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Assumptions

(V1) $V \in C^\infty(\mathbb{R}^3, \mathbb{R})$, V and its derivatives are uniformly bounded.

(V2) $\inf_{\mathbb{R}^3} V > 0$.

(V3) There exists $x_0 \in \mathbb{R}^3$ such that $\nabla V(x_0) = 0$.

(K1) $K \in C^\infty(\mathbb{R}^3, \mathbb{R})$, K and its derivatives are uniformly bounded.

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Theorem (I. Ianni, G. V.)

Let $p \in (1,5)$ and (V1), (V2), (V3), (K1), (K2) hold.
In addition, assume that

(V4) $x_0 \in \mathbb{R}^3$ **is a non-degenerate local minimum or maximum for** V, *namely $D^2V(x_0)$ is either positive or negative-definite.*

Then for $\epsilon > 0$ small, $(\mathcal{SP}')$ has a solution u_ϵ that concentrates at x_0.

Let for simplicity $x_0 = 0$ and $V(0) = 1$.

In $(\mathcal{SP}')$ we make a change of variable $x \longmapsto \epsilon x$, then we deal with the problem

$$-\Delta u + V(\epsilon x)u + \epsilon^2 K(\epsilon x)\phi_{\epsilon,u} u = |u|^{p-1}u, \qquad (\mathcal{SP}_\epsilon).$$

The solutions of $(\mathcal{SP}_\epsilon)$ are the critical points of the C^2- functional $I_\epsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\begin{aligned} I_\epsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(\epsilon x)u^2\right) dx + \frac{\epsilon^2}{4}\int_{\mathbb{R}^3} K(\epsilon x)\phi_{\epsilon,u} u^2\, dx \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx \end{aligned}$$

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Outline of the proofs

To prove the concentration result we have used a Perturbation Method, due to Ambrosetti and Badiale.

In other words we consider the functional I_ϵ as

$$I_\epsilon(u) = I_0(u) + G(\epsilon, u)$$

where the unperturbed functional $I_0(u)$, obtained for $\epsilon = 0$, is

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx$$

while the perturbation is

$$G(\epsilon, u) = \frac{1}{2}\int_{\mathbb{R}^3} [V(\epsilon x) - 1]u^2 dx + \frac{\epsilon^2}{4}\int_{\mathbb{R}^3} K(\epsilon x)\phi_{\epsilon,u} u^2 dx.$$

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$$I_\epsilon(u) = I_0(u) + G(\epsilon, u)$$

where the unperturbed functional $I_0(u)$, obtained for $\epsilon = 0$, is

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx$$

while the perturbation is

$$G(\epsilon, u) = \frac{1}{2}\int_{\mathbb{R}^3} [V(\epsilon x) - 1]u^2 dx + \frac{\epsilon^2}{4}\int_{\mathbb{R}^3} K(\epsilon x)\phi_{\epsilon,u} u^2 dx.$$

Outline of the proofs

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The critical points of the unperturbed problem are the solutions of the well-known problem

$$-\Delta u + u = |u|^{p-1}u, \qquad u \in H^1(\mathbb{R}^3)$$

which has a positive, ground state, solution $U \in H^1(\mathbb{R}^3)$, radially symmetric about the origin, unique up to translations, decaying exponentially, together its derivatives, as $|x| \to +\infty$.

Lyapunov-Schmidt reduction

We define the manifold of “approximate" solutions of the problem: fix $\bar{\xi} > 0$ and let

$$\mathcal{Z}_\epsilon := \left\{ z_\xi := U(\cdot - \xi) \quad : \quad \xi \in \mathbb{R}^3, \qquad |\xi| \leq \bar{\xi} \right\}.$$

Then for every $z_\xi \in \mathcal{Z}_\epsilon$, we define $W = \left(T_{z_\xi}\mathcal{Z}_\epsilon\right)^\perp$ and $P : H^1(\mathbb{R}^3) \to W$ the orthogonal projection onto W. Our approach is to find a pair $z_\xi \in \mathcal{Z}_\epsilon$, $w \in W$ such that $I'_\epsilon(z_\xi + w) = 0$, or equivalently:

$$\begin{cases} PI'_\epsilon(z_\xi + w) = 0, \\ \\ (Id - P)I'_\epsilon(z_\xi + w) = 0 \end{cases}$$

The fist equation above is called auxiliary equation, and the second one receives the name of bifurcation equation.

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Proposition

Consider a Hilbert space $\mathcal{H}$. Let $z \in \mathcal{H}$ and $T \in C^1(\mathcal{H}, \mathcal{H})$. Suppose that for some fixed $\delta > 0$, there holds:

(A1) $\|T(z)\|_{\mathcal{H}} \leq \delta$;

(A2) $T'(z) : \mathcal{H} \to \mathcal{H}$ is invertible and $\|(T'(z))^{-1}\|_{\mathcal{H}} \leq c$, $c > 0$;

Take $\rho \geq 2c$ and define:

$$B = \{u \in \mathcal{H} : \|u\|_{\mathcal{H}} \leq \rho\delta\}.$$

We further assume that

(A3) $\|T'(z+u) - T'(z)\|_{\mathcal{H}} < \frac{1}{\rho}$, $u \in B$.

Then there exists a unique $u \in B$ such that $T(z+u) = 0$.

The auxiliary Equation

First we find a solution $w \in W$ of the auxiliary equation proving

- $\|PI'_\epsilon(z_\xi)\| \leq C\epsilon^2$, $z_\xi \in \mathcal{Z}_\epsilon$;
- $PI''_\epsilon(z_\xi)$ is invertible and such that $\|[PI''_\epsilon(z_\xi)]^{-1}\| \leq \bar{C}$;
- $\|PI''_\epsilon(z_\xi + u) - PI''_\epsilon(z_\xi)\| \to 0$ for all $u \in B = \left\{w \in W : \|w\| \leq C_1\epsilon^2\right\}$.

Then there exists a solution $w = w_{\epsilon,z} \in W$ such that $\|w_{\epsilon,z}\| \leq C\epsilon^2$.

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The bifurcation equation

Now we find a solution for the bifurcation equation among the set of solutions of the auxiliary equation, which is:

$$\bar{\mathcal{Z}} = \left\{ z_\xi + w_{\epsilon, z_\xi} : z_\xi \in \mathcal{Z}_\xi \right\}.$$

By the Implicit Function Theorem it is easy to check that $\bar{\mathcal{Z}}$ is a C^1 manifold. Moreover, it is well-known that $\bar{\mathcal{Z}}$ is a natural constraint for I_ϵ for ϵ small. In other words, critical points of $I_\epsilon|_{\bar{\mathcal{Z}}}$ are solutions of the bifurcation equation, and hence solutions of $(\mathcal{SP}_\epsilon)$.

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The reduced functional

So, let us define the reduced functional as the restriction of the functional I_ϵ to the natural constraint $\bar{\mathcal{Z}}$, namely $\Phi_\epsilon : B_{\bar{\xi}}(0) \subset \mathbb{R}^3 \to \mathbb{R}$,

$$\Phi_\epsilon(\xi) = I_\epsilon(z_\xi + w_{\epsilon,z_\xi})$$

We look for critical points of Φ_ϵ.
Using the information on $\|w_{\epsilon,z_\xi}\|$, we will be able to find an expansion of $\Phi_\epsilon(\xi)$.

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Expansion in the non-degenerate case

Proposition (non-degenerate case)

$$\Phi_\epsilon(\xi) = C_0 + \epsilon^2 \Gamma_1(\xi) + o(\epsilon^2), \qquad \text{for } |\xi| \le \bar{\xi}$$

where

$$\begin{aligned}
C_0 &= I_0(U); \\
\Gamma_1(\xi) &= C_1 + C_2 \langle D^2 V(0)\xi, \xi\rangle; \\
C_1 &= \frac{1}{4}\int_{\mathbb{R}^3} \langle D^2 V(0)x, x\rangle U^2(x)dx + \frac{K(0)^2}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U^2(y)U^2(x)}{|x-y|}dydx; \\
C_2 &= \frac{1}{4}\int_{\mathbb{R}^3} U^2(x)dx.
\end{aligned}$$

Lemma

$$\Phi_\epsilon(\xi) = C_0 + \epsilon^\beta \Gamma(\xi) + o(\epsilon^\beta), \qquad |\xi| \le \bar{\xi}$$

and assume that $\xi = 0$ is a non-degenerate minimum (or maximum) for Γ. Then Φ_ϵ has a minimum (or maximum) in some ξ_ϵ such that $\xi_\epsilon \to 0$ as $\epsilon \to 0$.

In conclusion, recalling the change of variable

$$v_\epsilon(x) := u_\epsilon\left(\frac{x}{\epsilon}\right) \sim z_{\xi_\epsilon}\left(\frac{x}{\epsilon}\right) = U\left(\frac{x}{\epsilon} - \xi_\epsilon\right),$$

is a solution of $(\mathcal{SP}')$ which concentrates near the critical point 0.

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A Multiplicity Result

By using the same technique outlined before one can also infer the existence of multiple solutions.
Let, for simplicity, the problem

$$-\epsilon^2 \Delta u + V(x)u + \phi_u u = |u|^{p-1}, \qquad u \in H^1(\mathbb{R}^3).$$

If V has a finite collection of non-degenerate critical points x_i, then we obtain a spike solution around each critical point.
However, the bumps are well separated, namely the effect of one bump on another bump is neglected.

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Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of the potential,
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Cluster Solutions

In a work of X. Kang and J. Wei, the authors consider the nonlinear Schrödinger equation

$$-\epsilon^2 \Delta u + V(x)u = |u|^{p-1}u, \qquad x \in \mathbb{R}^3$$

proving the existence of a cluster solution around a local maximum of V and non-existence of a cluster solution around a local minimum of V.

Case of Schrödinger-Poisson problem

Our problem is now:

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Assumptions

(V1) V has a local strict minimum point in P_0, namely there exists a bounded open set $\mathcal{U}$ such that $P_0 \in \mathcal{U}$ and

$$V(P_0) = \min_{x \in \bar{\mathcal{U}}} V(x) < V(P), \qquad \forall\, P \in \mathcal{U} \setminus \{P_0\}$$

Up to a translation and dilatation, we can assume $P_0 = 0$, $V(0) = 1$.

(V2) $V(x) = 1 + |g(x)|^\alpha$ for any $x \in \mathcal{U}$, where $g : \mathcal{U} \to \mathbb{R}$ is a $C^{2,1}$ function and $\alpha > 2$.

In particular, there holds:

(V2') $V(x) \leq 1 + C|x|^\alpha$ for $x \in \mathcal{U}$ and some $C > 0$.

Remark

Observe that under the above conditions the local minimum must be degenerate. We point out that conditions (V1)-(V2') are sufficient for most of our arguments. We need condition (V2) for technical reasons, to be able to rule out possible undesired oscillations of the derivatives of V near 0.

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Theorem (D. Ruiz, G. V.)

Assume that V satisfies (V1) and (V2) and suppose $p \in (1,5)$. Then for any positive integer $K \in \mathbb{Z}$, there exists $\epsilon_K > 0$ such that for any $\epsilon < \epsilon_K$ there exists a positive solution u_ϵ of $(\mathcal{SP}')$ with K bumps converging to 0. More specifically, there exists $Q_1^\epsilon, \dots Q_k^\epsilon \in \mathbb{R}^3$ such that:

1. $Q_i^\epsilon \to 0$, $\epsilon^{-1}|Q_i^\epsilon| \to +\infty$ *as* $\epsilon \to 0$.
2. *Defining* $\tilde{u}_\epsilon(x) = u_\epsilon(\epsilon x)$, *we have that* $\tilde{u}_\epsilon(x) = \sum_{i=1}^K U(x - \epsilon^{-1} Q_i^\epsilon) + o(1)$, *as* $\epsilon \to 0$.

The Lyapunov-Schmidt reduction will be made, in this case, around an appropriate set of “approximating solutions".

For any $K \in \mathbb{N}$, we define

$$\Lambda_\epsilon = \Big\{ \mathbf{P} \in \mathbb{R}^{3K} : |P_i - P_j| \geq \epsilon^{\frac{2-\alpha}{\alpha+1}+\delta},\ i \neq j, \\ V(\epsilon P_i) \leq 1 + \epsilon^{\frac{3\alpha}{\alpha+1}-\delta},\ \epsilon P_i \in \mathcal{U} \Big\}$$

where $\delta > 0$ is chosen small enough so that $\frac{3\alpha}{\alpha+1} - \delta > 2$ (this is possible since $\alpha > 2$). Observe that $\frac{2-\alpha}{\alpha+1} + \delta < 0$ and Λ_ϵ is not empty for ϵ small enough.

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Fix $\mathbf{P} = (P_1, ..., P_K) \in \Lambda_\epsilon$. Setting $z_{P_i}(x) = U(x - P_i)$, we define the manifold of "approximate solutions":

$$\mathcal{Z} = \left\{ z_{\mathbf{P}}(x) = \sum_{i=1}^{K} z_{P_i}(x) \; : \; \mathbf{P} \in \Lambda_\epsilon \right\}.$$

We first prove the existence of a solution of the auxiliary equation, then we find an expansion for the reduced functional.

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The reduced functional

$$\Phi_\epsilon(\mathbf{P}) = C_0 + \epsilon^2 C_1 + C_2 \sum_{i=1}^{K} V(\epsilon P_i) + C_3 \epsilon^2 \sum_{i \neq j} \frac{1}{|P_i - P_j|} + o(\epsilon^{\frac{3\alpha}{\alpha+1} - \delta}). \quad (1)$$

Proposition

For ϵ sufficiently small, the following minimization problem

$$\min \{\Phi_\epsilon(\mathbf{P}) \ : \ \mathbf{P} \in \Lambda_\epsilon\} \quad (2)$$

has a solution $\mathbf{P}_\epsilon \in \Lambda_\epsilon$.

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Let us consider the problem

$$\begin{cases} -\Delta u + u + K(x)\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \qquad (\mathcal{SP})$$

where $p \in (1,5)$ and $K : \mathbb{R}^3 \to \mathbb{R}$ is a non-negative bounded function.
We assume that K is a radial function, that is $K(x) = K(|x|) = K(r)$ satisfying the following condition:

(K) There are constants $a > 0$, $m > \frac{3}{2}$, $\theta > 0$ such that

$$K(r) = \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right),$$

as $r \to +\infty$.

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as $r \to +\infty$.

Again, the problem $(\mathcal{SP})$ can be reduced into a single equation:

$$-\Delta u + u + K(|x|)\phi_u u = |u|^{p-1}u, \qquad u \in H^1(\mathbb{R}^3), \qquad (\mathcal{SP}')$$

Theorem

If $K(x)$ satisfies (K), then the problem $(\mathcal{SP}')$ has infinitely many non-radial positive solutions.

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Theorem

If $K(x)$ satisfies (K), then the problem $(\mathcal{SP}')$ has infinitely many non-radial positive solutions.

To prove the Theorem we construct solutions with large number of bumps near infinity.

In fact, since $K(r) \to 0$ as $r \to +\infty$, the solutions of $(\mathcal{SP}')$ can be approximated by using the solution U of the limit problem

$$\begin{cases} -\Delta u + u = u^p, & \text{in } \mathbb{R}^3, \\ u > 0, & \text{in } \mathbb{R}^3, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{3}$$

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For any positive integer k, let us define

$$P_j = \left(r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right) \in \mathbb{R}^3, \qquad j = 1, \dots, k,$$

with $r \in S_k := \left[\left(\frac{m}{\pi} - \beta \right) k \log k, \left(\frac{m}{\pi} + \beta \right) k \log k \right]$ for some $\beta > 0$ sufficiently small and

$$z_r(x) = \sum_{j=1}^{k} U_{P_j}(x),$$

where $U_{P_j}(\cdot) := U(\cdot - P_j)$.

If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we set

$$H_s = \left\{ u \in H^1(\mathbb{R}^3) \,\middle|\, \begin{array}{l} u \text{ is even in } x_2, x_3; \\ u(r\cos\theta, r\sin\theta, x_3) = \\ = u\left(r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), x_3\right) \\ j = 1, \dots, k-1 \end{array} \right\}.$$

We remark that if $u \in H_s$, then $\phi_u \in D_s$, where

$$D_s = \left\{ \phi \in D^{1,2}(\mathbb{R}^3) \,\middle|\, \begin{array}{l} \phi \text{ is even in } x_2, x_3; \\ \phi(r\cos\theta, r\sin\theta, x_3) = \\ = \phi\left(r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), x_3\right) \\ j = 1, \dots, k-1 \end{array} \right\};$$

Finally, let us define

$$\Omega_j := \left\{ x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \;:\; \langle \frac{x'}{|x'|}, \frac{P_j}{|P_j|} \rangle \geq \cos\frac{\pi}{k} \right\}.$$

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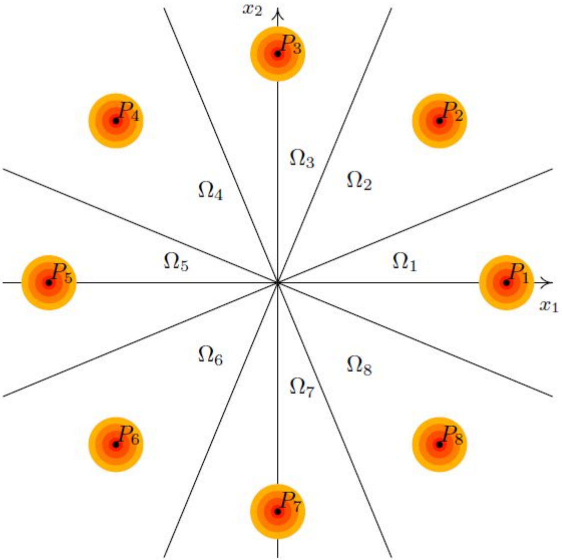

Figure: Position of the multi-bumps solutions

Lyapunov-Schmidt reduction

The manifold of the approximate solutions is now given by

$$\mathcal{Z} := \{z_r \ : \ r \in S_k\}$$

First we find the solution of the auxiliary equation w.
Then we study the remaining finite dimensional equation.
In this case the reduced functional is given by

$$F(r) = k\left[C_0 + \frac{B_1}{r^{2m}} + \frac{B_2 k \log k}{r^{2m+1}} - B_3 \sum_{i=2}^{k} \int_{\mathbb{R}^3} U_{P_1}^p U_{P_i}\, dx + O\left(\frac{1}{k^{2m+\sigma}}\right)\right],$$

The problem

$$\max\{F(r) : r \in S_k\}$$

has a solution since F is continuous on a compact set.
Then we show that this maximum, r_k, is an interior point of S_k.
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P. d'Avenia, A. Pomponio, G. V.,
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Existence of Ground and Bound States for $(\mathcal{SP})$

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Dealing with I, one has to face various difficulties:

a) The competing effect of the nonlocal term with the nonlinear term gives rise to very different situations as p varies in the interval $(1, 5)$;

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Assumptions

Let $p \in (3,5)$.

Moreover we assume that $a(x)$ and $K(x)$ verify, respectively

$$\text{(a1)} \qquad \lim_{|x|\to+\infty} a(x) = a_\infty > 0, \quad \alpha(x) := a(x) - a_\infty \in L^{\frac{6}{5-p}}(\mathbb{R}^3);$$

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The study is carried out considering I constrained on its Nehari manifold,

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On $\mathcal{N}$ I turns out to be bounded from below by a positive constant.

Let us define

$$m := \inf\{I(u) \, : \, u \in \mathcal{N}\} > 0. \tag{4}$$

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Some Remarks

Since $K(x) \to 0$ and $a(x) \to a_\infty$ as $|x| \to +\infty$, the problem at infinity is given by

$$-\Delta u + u = |u|^{p-1}u, \qquad u \in H^1(\mathbb{R}^3) \qquad (\mathcal{P}_\infty)$$

The solutions of $(\mathcal{P}_\infty)$ are the critical points of the functional $I_0 : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

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$$\begin{aligned} &\text{a) } I(u_n) \text{ is bounded;} \\ &\text{b) } \nabla I_{|\mathcal{N}}(u_n) \to 0 \text{ strongly in } H^1(\mathbb{R}^3). \end{aligned} \quad (6)$$

Then replacing $(u_n)_n$, if necessary, with a subsequence, there exist a solution $\bar{u}$ of $(\mathcal{SP}')$, a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, ..., u^k$ of $H^1(\mathbb{R}^3)$ and k sequences of points (y_n^j), $y_n^j \in \mathbb{R}^3$, $0 \le j \le k$ such that

$$\begin{aligned} &\text{(i) } |y_n^j| \to +\infty, \ |y_n^j - y_n^i| \to +\infty \text{ if } i \neq j, \ n \to +\infty; \\ &\text{(ii) } u_n - \sum_{j=1}^{k} u^j(\cdot - y_n^j) \longrightarrow \bar{u}, \text{ in } H^1(\mathbb{R}^3); \\ &\text{(iii) } I(u_n) \to I(\bar{u}) + \sum_{j=1}^{k} I_0(u^j); \\ &\text{(iv) } u^j \text{ are non trivial weak solutions of } (\mathcal{P}_\infty). \end{aligned} \quad (7)$$

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Then replacing $(u_n)_n$, if necessary, with a subsequence, there exist a solution $\bar{u}$ of $(\mathcal{SP}')$, a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, \ldots, u^k$ of $H^1(\mathbb{R}^3)$ and k sequences of points (y_n^j), $y_n^j \in \mathbb{R}^3$, $0 \le j \le k$ such that

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Corollary

Let $(u_n)_n$ be a $(PS)_d$ sequence. Then $(u_n)_n$ is relatively compact for all $d \in (0, m_\infty)$.
Moreover, if $I(u_n) \to m_\infty$, then either $(u_n)_n$ is relatively compact or the statement of previous Theorem holds with $k = 1$, and $u^1 = U$ (U ground state of $(\mathcal{P}_\infty)$).

If we assume

(a2) $a(x) \geq a_\infty \quad \forall x \in \mathbb{R}^3, \quad a(x) - a_\infty > 0$ on a positive measure set,

the problem can be faced by a minimization argument.

When (a2) holds, the problem

$$-\Delta u + u = a(x)|u|^{p-1}u$$

admits a ground state solution, that is denoted by w_a and whose energy is

$$m_a = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w_a\|^2 < m_\infty.$$

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Theorem

Let (a1), (a2), (K) hold. Furthermore assume either

$$|K|_2^2 < \frac{m_\infty^\vartheta - m_a{}^\vartheta}{\sigma m_a^{1+\vartheta}}, \tag{8}$$

with $\vartheta = \frac{p-3}{p+1}$ *and* $\sigma = \frac{2(p+1)}{p-1}\ \bar{S}^{-2}S^{-4}$, *or*

$$\int_{\mathbb{R}^3} K(x)\phi_U U^2 dx < \frac{4}{p+1}\int_{\mathbb{R}^3} \alpha(x)|U|^{p+1}dx. \tag{9}$$

Then the problem $(\mathcal{SP}')$ *has a positive ground state solution.*

Corollary

The functional I satisfies the $(PS)_d$ condition for all $d \in (m_\infty, 2m_\infty)$

On the contrary when

$$(a3)\;\; a(x) \leq a_\infty \quad \forall x \in \mathbb{R}^3, \quad \mathcal{A} := \inf_{\mathbb{R}^3} a(x) > 0,$$

holds, the infimum of I on $\mathcal{N}$ cannot be achieved and the existence of a solution is a more delicate question that is handled by using the notion of barycenter to build a min-max level belonging to an interval of the values of I in which the compactness holds.

Idea to get positive bound states

First we define the barycenter of a function $u \in H^1(\mathbb{R}^3)$, $u \neq 0$.

Setting

$$\mu(u)(x) = \frac{1}{|B_1(0)|}\int_{B_1(x)} |u(y)|dy, \qquad \mu(u) \in L^\infty(\mathbb{R}^3) \text{ and is continuous}$$

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2}\max \mu(x)\right]^+, \qquad \hat{u} \in C_0(\mathbb{R}^3);$$

we define the barycenter $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ by

$$\beta(u) = \frac{1}{|\hat{u}|_1}\int_{\mathbb{R}^3} x\hat{u}(x)dx \in \mathbb{R}^3.$$

Since $\hat{u}$ has compact support, β is well defined. Moreover the following properties hold:

1. β is continuous in $H^1(\mathbb{R}^3) \setminus \{0\}$;
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Let us define

$$b_0 := \inf\{I(u) \;\;:\;\; u \in \mathcal{N},\;\; \beta(u) = 0\}. \tag{10}$$

Lemma

$b_0 > m$.

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We define the operator:

$$\Gamma : \mathbb{R}^3 \to \mathcal{N}$$

as

$$\Gamma[z](x) = t_z U(x - z)$$

where U is the positive solution of $(\mathcal{P}_\infty)$ and t_z is chosen such that $\Gamma[z] \in \mathcal{N}$.

Lemma

$$\lim_{|z|\to+\infty} I(\Gamma(z)) = m_\infty .$$

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Lemma

Assume that

$$\frac{1+\eta|K|_2^2}{\mathcal{A}} < 2^{\frac{p-3}{p+1}} \tag{11}$$

with $\eta = \frac{2(p+1)}{p-1} S^{-4} \bar{S}^{-2} m_\infty$, hold. Then

$$I(\Gamma[z]) < 2m_\infty. \tag{12}$$

Theorem

Let (a1), (a3), (K) hold. Furthermore assume

$$\frac{1+\eta|K|_2^2}{\mathcal{A}} < 2^{\frac{p-3}{p+1}} \tag{13}$$

hold. Then the problem $(\mathcal{SP}')$ *has (at least) one positive solution.*

Proof

By the previous Lemmas, there exists $\bar{\rho} > 0$ such that for all $\rho \geq \bar{\rho}$

$$m_\infty < \max_{|z|=\rho} I(\Gamma[z]) < b_0. \tag{14}$$

In order to apply the Linking Theorem we take

$$Q = \Gamma(\bar{B}_{\bar{\rho}}(0)), \quad S = \{u \in \mathcal{N} \ : \ \beta(u) = 0\}.$$

We claim that S and ∂Q links, that is

$$\begin{aligned} &\text{a) } \partial Q \cap S = \emptyset; \\ &\text{b) } h(Q) \cap S \neq \emptyset \ \forall \ h \in \mathcal{H} = \{h \in \mathcal{C}(Q, \mathcal{N}) : \ h_{|_{\partial Q}} = id\} \end{aligned} \tag{15}$$

hold.

Proof

(15)-a) follows at once, observing that if $u \in \partial Q$ then $u = \Gamma[\bar{z}]$, $|\bar{z}| = \bar{\rho}$, and, by the properties of the barycenter map we get $\beta(u) = \beta(\Gamma[\bar{z}]) = \bar{z}$. To verify (15)-b), let consider $h \in \mathcal{H}$ and define

$$T : \bar{B}_{\bar{\rho}}(0) \to \mathbb{R}^3, \qquad T(z) = \beta \circ h \circ \Gamma[z].$$

T is a continuous function, and, for all $|z| = \bar{\rho}$, $\Gamma[z] \in \partial Q$, hence $h \circ \Gamma[z] = \Gamma[z]$ that implies $T(z) = z$. By the Brower fixed point theorem there exists $z \in B_{\bar{\rho}}(0)$ such that $T(z) = 0$ and this means that $h(\Gamma[z]) \in S$. Therefore $h(Q) \cap S \neq \emptyset$.

Proof

Now (14) can be written as $b_0 = \inf_S I > \max_{\partial Q} I$. Let us define

$$d := \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)).$$

Then by (15)- b), $d \geq b_0 > m \equiv m_\infty$. Moreover, taking $h = id$ and using Lemma 3 we deduce $d < 2m_\infty$. Since, by Lemma 2, (PS) holds in $(m_\infty, 2m_\infty)$, by the Linking theorem d is a critical value of I.
This proves the existence of a non trivial solution of (SP').

G. Cerami, G. V.,
Positive solutions for some non autonomous Schrödinger-Poisson Systems
Journal of Differential Equations **248**, *no. 3 (2010), 521–543.*

G. V.,
Ground states for Schrödinger-Poisson type systems
to appear on Nonlinear Differential Equations

G. V.,
Bound states for Schrödinger-Newton type systems
preprint

Let us consider the problem

$$-\Delta u + u - K(x)\phi_u u = a(x)|u|u, \qquad (\mathcal{SN})$$

assuming that

$$\text{(a1)} \qquad \lim_{|x|\to+\infty} a(x) = a_\infty > 0, \quad \alpha(x) := a(x) - a_\infty \in L^{\frac{6}{5-p}}(\mathbb{R}^3);$$

$$\mathcal{A} := \inf_{\mathbb{R}^3} a(x) > 0;$$

$$\text{(K1)} \qquad \lim_{|x|\to+\infty} K(x) = K_\infty > 0, \quad \eta(x) := K(x) - K_\infty \in L^2(\mathbb{R}^3);$$

$$\mathcal{K} := \inf_{\mathbb{R}^3} K(x) > 0.$$

The problem at infinity

$$-\Delta u + u - K_\infty \tilde{\phi}_u u = a_\infty |u| u, \qquad (\mathcal{SN}_\infty)$$

Proposition

The problem $(\mathcal{SN}_\infty)$ has a positive radial ground state $\bar{w}$.

Let

$$\bar{c} = I_\infty(\bar{w}) = \frac{1}{2}\|\bar{w}\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{w}} \bar{w}^2 \, dx - \frac{1}{3}\int_{\mathbb{R}^3} a_\infty |\bar{w}|^3 \, dx$$

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It is clear that one can infer the existence of ground states solution for $(\mathcal{SN})$ under particular assumptions on $K(x)$ and $a(x)$.

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PROBLEM: Establish the existence of a bound state solution for $(\mathcal{SN})$.

Existence of Ground States

It is clear that one can infer the existence of ground states solution for $(\mathcal{SN})$ under particular assumptions on $K(x)$ and $a(x)$.
PROBLEM: Establish the existence of a bound state solution for $(\mathcal{SN})$.

All the positive solutions of $(\mathcal{SN}_\infty)$ are radially symmetric

Theorem

The positive solutions of $(\mathcal{SN}_\infty)$ *must be radially symmetric and monotone decreasing about some fixed point.*

The key step to prove the Theorem is to transform the differential equation $(\mathcal{SN}_\infty)$ into an integral system by virtue of the Bessel potentials.

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The key step to prove the Theorem is to transform the differential equation $(\mathcal{SN}_\infty)$ into an integral system by virtue of the Bessel potentials.

Bessel Potential

The Bessel potential $\mathcal{G}_2 = (Id - \Delta)^{-1}$ can be seen as the inverse operator of the positive operator $Id - \Delta$ in the Sobolev space $H^1(\mathbb{R}^3)$.
For convenience, the Bessel potential is usually expressed in the convolution form

$$\mathcal{G}_2(f) = g_2 * f,$$

in which the Bessel kernel g_2 can be determined explicitly by

$$g_2(x) = \frac{1}{(4\pi)\Gamma(1)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{-1/2} \frac{d\delta}{\delta}.$$

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Hence we can transform the differential equation $(\mathcal{SN}'_\infty)$ into an integral equation involving the Bessel potential $\mathcal{G}_2$. Indeed,

$$\begin{aligned} u &= (-\Delta+1)^{-1}\left(K_\infty\tilde{\phi}_u u + a_\infty u^2\right) \\ &= (-\Delta+1)^{-1}\left[K_\infty^2\left(\frac{1}{|x|}*u^2\right)u + a_\infty u^2\right] \\ &= g_2 * \left[K_\infty^2\left(\frac{1}{|x|}*u^2\right)u + a_\infty u^2\right] \end{aligned}$$

or equivalently

$$\begin{cases} u = g_2 * \left(K_\infty^2 vu + a_\infty u^2\right) \\ \\ v = \dfrac{1}{|x|} * u^2. \end{cases} \tag{16}$$

The most useful fact concerning Bessel potentials is that it can be employed to characterize the Sobolev space $W^{k,p}(\mathbb{R}^3)$.

Indeed we have that for all $p \in (1, +\infty)$ that

$$\mathcal{G}_2(f) = g_2 * f \in W^{2,p}(\mathbb{R}^3), \qquad \forall\, f \in L^p(\mathbb{R}^3).$$

By the Sobolev embedding, we obtain the estimate

$$|\mathcal{G}_2(f)|_q \le C_{r,s,3}|f|_s, \quad \forall\, f \in L^s(\mathbb{R}^3), \tag{17}$$

in which $0 \le \frac{1}{s} - \frac{2}{3} \le \frac{1}{q} \le \frac{1}{s}$. The estimate (17) will be very useful in our arguments below.

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Proof

For a given real number λ, let us define

$$\Sigma_\lambda := \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 \geq \lambda\right\},$$

$$\Sigma_\lambda^u := \{x \in \Sigma_\lambda \ : \ u_\lambda(x) > u(x)\},$$

$$\Sigma_\lambda^v := \{x \in \Sigma_\lambda \ : \ v_\lambda(x) > v(x)\},$$

and we denote by $x^\lambda = (2\lambda - x_1, x_2, x_3)$ the reflected point with respect to the plane $\{x_1 = \lambda\}$ and denote $u_\lambda(x) = u(x^\lambda)$ and $v_\lambda(x) = v(x^\lambda)$.

Decomposition of $u_\lambda - u$ and of $v_\lambda - v$

For any positive solution of $(\mathcal{SN}_\infty)$, we have for all $x \in \mathbb{R}^3$ that

$$
\begin{aligned}
u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} \Big(g_2(x-y) - g_2(x^\lambda - y) \Big) \left[K_\infty^2 (v_\lambda u_\lambda - vu) \right] dy \\
&\quad + \int_{\Sigma_\lambda} \Big(g_2(x-y) - g_2(x^\lambda - y) \Big) \left[a_\infty (u_\lambda^2 - u^2) \right] dy.
\end{aligned}
$$

$$
v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|} - \frac{1}{|x^\lambda - y|} \right) (u_\lambda^2(y) - u^2(y)) dy.
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Proof

Step 1: By using the decomposition of $u_\lambda - u$ we find for all $x \in \Sigma_\lambda$:

$$
\begin{aligned}
|u_\lambda - u|_{2,\Sigma_\lambda^u} \leq\ & \bar{C}_1 \cdot |v_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} + \bar{C}_2 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |v_\lambda - v|_{6,\Sigma_\lambda^v} \\
& + \bar{C}_3 \cdot |u_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u}. \qquad (18)
\end{aligned}
$$

Similarly, from the decomposition of $v_\lambda - v$, we obtain, for all $x \in \Sigma_\lambda$, that

$$
v_\lambda(x) - v(x) \leq 2 \int_{\Sigma_\lambda^u} \frac{1}{|x-y|} u_\lambda(y)(u_\lambda(y) - u(y))dy. \qquad (19)
$$

By the Hardy-Littlewood-Sobolev inequality, we deduce from (19) that

$$
|v_\lambda - v|_{6,\Sigma_\lambda^v} \leq C_4 |u_\lambda(u_\lambda - u)|_{\frac{6}{5},\Sigma_\lambda^v} \leq \bar{C}_4 |u_\lambda|_{3,\Sigma_\lambda^v} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^v}. \qquad (20)
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Proof

Step 1: By using the decomposition of $u_\lambda - u$ we find for all $x \in \Sigma_\lambda$:

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Proof

Step 2: We show that for sufficient negative values of λ, the set Σ_λ^u and Σ_λ^v must be empty. In fact, the estimates above imply

$$\begin{aligned} |u_\lambda - u|_{2,\Sigma_\lambda^u} \leq \; & \bar{C}_1 \cdot |v_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} + \bar{C}_5 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |u_\lambda|_{3,\Sigma_\lambda^v} \cdot |u_\lambda - u|_2 \\ & + \bar{C}_3 \cdot |u_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u}. \end{aligned}$$

We can choose N sufficiently large such that for $\lambda \leq -N$, we have

$$\bar{C}_1 |v_\lambda|_{6,\Sigma_\lambda^u} \leq \frac{1}{6}, \quad \bar{C}_5 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |u_\lambda|_{3,\Sigma_\lambda^v} \leq \frac{1}{6}, \quad \bar{C}_3 |u_\lambda|_{6,\Sigma_\lambda^u} \leq \frac{1}{6},$$

which implies that

$$|u_\lambda - u|_{2,\Sigma_\lambda^u} = 0,$$

and therefore Σ_λ^u must be measure zero and hence empty.
Then also $\Sigma_\lambda^v = \emptyset$.

Proof

Step 3: Now we have that for $\lambda \leq -N$

$$u(x) \geq u_\lambda(x), \qquad \forall\ x \in \Sigma_\lambda. \tag{21}$$

Thus we can start moving the plane $\{x_1 = \lambda\}$ continuously from $\lambda \leq -N$ to the right as long as (21) holds. Suppose that at a λ_0 we have $u \geq u_{\lambda_0}$ on Σ_{λ_0}, but $u \not\equiv u_{\lambda_0}$ on Σ_{λ_0}, we will show that the plane can be moved further to the right.

More precisely, we prove that there exists an ϵ depending on the solution u itself such that $u \geq u_\lambda$ on Σ_λ for all λ in $[\lambda_0, \lambda_0 + \epsilon)$.

Proof

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Proof

By using the decomposition of $u_\lambda - u$ and of $v_\lambda - v$ and moreover by using the estimates of their norm, we prove that when moving plane process stops, we must have $u \equiv u_{\lambda_0}$, and $u_\lambda \leq u$ on Σ_λ when $\lambda < \lambda_0$.
By a translation, we may assume that $u(0) = \max_{x\in\mathbb{R}^3} u(x)$. Then it follows that the moving plane process from any direction must stop at the origin. Hence u must be radially symmetric and monotone decreasing in the radial direction.

Uniqueness of the radial solution of $(\mathcal{SN}_\infty)$

Let us first recall the following theorem known as Newton's Theorem.

Theorem

For any radial function $\rho = \rho(|x|) \in L^1(\mathbb{R}^3, (1+|x|)^{-1}dx)$, *we have*

$$\left(|x|^{-1} * \rho\right)(r) = V(\rho) - F_\rho(r)$$

where

$$V(\rho) = \int_{\mathbb{R}^3} \frac{\rho(|x|)}{|x|} dx, \qquad F_\rho(r) = 4\pi \int_0^r s\left(1 - \frac{s}{r}\right)\rho(s)\, ds.$$

Since all positive solutions of $(\mathcal{SN}_\infty)$are radial, we have to show the uniqueness of the radial solution of $(\mathcal{SN}_\infty)$. By using Newton Theorem, $(\mathcal{SN}_\infty)$ can be transformed into

$$-\Delta u + K_\infty^2 F_{u^2} u - a_\infty u^2 = \mu u, \tag{22}$$

where $\mu := K_\infty^2 V(u^2) - 1$. It is possible to show that $\mu > 0$. We set

$$A(u) := K_\infty^2 F_{u^2} - a_\infty u,$$

then (22) becomes

$$-\Delta u + A(u)u = \mu u. \tag{23}$$

Proposition

The problem (23) has a unique radial positive solution provided $\frac{a_\infty}{K_\infty^2}$ is sufficiently small.

Non-degeneracy condition

Theorem

Let $(v, \psi) \in H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ *be a solution of*

$$\begin{cases} \Delta v - v + K_\infty \tilde{\phi}_w v + K_\infty w \psi + 2a_\infty w v = 0 \\ \Delta \psi + K_\infty v w = 0. \end{cases} \tag{24}$$

where $(w, \tilde{\phi}_w)$ *is the solution of* $(\mathcal{SN}_\infty)$. *Then*

$$(v, \psi) \in \text{span}\left\{ \frac{\partial(w, \tilde{\phi}_w)}{\partial x_j}\,;\ j = 1, 2, 3 \right\}.$$

Remark

Suppose that $v \in H^2(\mathbb{R}^3)$ satisfies the following problem

$$\Delta v - v + K_\infty \tilde{\phi}_w v + K_\infty w \int_{\mathbb{R}^3} \frac{K_\infty v(y) w(y)}{|x-y|} \, dy + 2a_\infty w v = 0,$$

then by Theorem 10 it follows that

$$v \in \operatorname{span} \left\{ \frac{\partial w}{\partial x_j} \; : \; j = 1, 2, 3. \right\}.$$

Theorem

Let (a1)-(K1) and

(H) $K(x) \leq K_\infty$; $a(x) \leq a_\infty$ *for all* $x \in \mathbb{R}^3$ *and* $a_\infty - a(x) > 0$ *on a positive measure set;*

hold. Then there exists (at least) one positive bound state solution of $(\mathcal{SN})$ *provided* $\dfrac{\max\{K_\infty^2, a_\infty\}}{\min\{\mathcal{K}^2, \mathcal{A}\}}$ *is sufficiently small.*

Thank you for the attention!