Elliptic equations with measure data

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Contents

CHAPTER 1

Existence with regular data in the linear case

Before stating and proving the existence theorem for linear elliptic equations, we need some tools.

1. Minimization in Banach spaces

Let E be a Banach space, and let $J : E \to \mathbb{R}$ be a functional.

DEFINITION 1.1. A functional $J : E \to \mathbb{R}$ is said to be *weakly lower semicontinuous* if

$$u_n \rightharpoonup u \quad \Rightarrow \quad J(u) \leq \liminf_{n \to +\infty} J(u_n).$$

DEFINITION 1.2. A functional $J : E \to \mathbb{R}$ is said to be *coercive* if

$$\lim_{\|u\|_E \to +\infty} J(u) = +\infty.$$

EXAMPLE 1.3. If $E = \mathbb{R}$, the function $J(x) = x^2$ is an example of a (weakly) lower semicontinuous and coercive functional. Another example is $J(u) = \|u\|_E$.

THEOREM 1.4. *Let E be a reflexive Banach space, and let $J : E \to \mathbb{R}$ be a coercive and weakly lower semicontinuous functional (not identically equal to $+\infty$). Then J has a minimum on E.*

Proof. Let

$$m = \inf_{v \in E} J(v) < +\infty,$$

and let $\{v_n\}$ in E be a minimizing sequence, i.e., v_n is such that

$$\lim_{n \to +\infty} J(v_n) = m.$$

We begin by proving that $\{v_n\}$ is bounded. Indeed, if it were not, there would be a subsequence $\{v_{n_k}\}$ such that

$$\lim_{k \to +\infty} \|v_{n_k}\| = +\infty.$$

Since J is coercive, we will have

$$m = \lim_{n \to +\infty} J(v_n) = \lim_{k \to +\infty} J(v_{n_k}) = +\infty,$$

which is false. Therefore, $\{v_n\}$ is bounded in E and so, being E reflexive, there exists a subsequence $\{v_{n_k}\}$ and an element v of E such that v_{n_k} weakly converges to v as k diverges. Since J is weakly lower semicontinuous, we have

$$m \leq J(v) \leq \liminf_{k\to+\infty} J(v_{n_k}) = \lim_{n\to+\infty} J(v_n) = m,$$

so that v is a minimum of J. □

2. Hilbert spaces

2.1. *Linear forms and dual space.* We recall that a Hilbert space H is a vector space where a scalar product $(\cdot|\cdot)$ is defined, which is complete with respect to the distance induced by the scalar product by the formula

$$d(x,y) = \sqrt{(x-y|x-y)}.$$

Examples of Hilbert spaces are $\mathbb{R}$ (with $(x|y) = x\,y$), $\mathbb{R}^N$ (with the "standard" scalar product), ℓ^2, and $L^2(\Omega)$ with

$$(f|g) = \int_\Omega f\,g.$$

THEOREM 1.5 (Riesz). *Let H be a separable Hilbert space, and let T be an element of its dual H', i.e., a linear application $T : H \to \mathbb{R}$ such that there exists $C \geq 0$ such that*

$$|\langle T, x\rangle| \leq C\|x\|, \quad \forall x \in H. \tag{1.1}$$

Then there exists a unique y in H such that

$$\langle T, x\rangle = (y|x), \quad \forall x \in H.$$

Proof. Denote by $\{e_h\}$ a complete orthonormal system in H, i.e. a sequence of vectors of H such that $(e_h|e_k) = \delta_{hk}$, and such that, for every x in H, one has

$$x = \sum_{h=1}^{+\infty} (x|e_h)e_h.$$

It is then well known that there exists a bijective isometry $\mathcal{F}$ from H to ℓ^2, defined by $\mathcal{F}(x) = \{(x|e_h)\}$. We claim that $\{\langle T, e_h\rangle\}$ belongs to ℓ^2. Indeed, if

$$y_n = \sum_{h=1}^{n} \langle T, e_h\rangle e_h,$$

we have, by linearity and by (1.1),

$$\sum_{h=1}^{n}(\langle T, e_h\rangle)^2 = \langle T, y_n\rangle \leq C\|y_n\| = C\left(\sum_{h=1}^{n}(\langle T, e_h\rangle)^2\right)^{\frac{1}{2}},$$

so that

$$\sum_{h=1}^{n}(\langle T, e_h\rangle)^2 \leq C^2,$$

which yields (letting n tend to infinity) that $\{\langle T, e_h\rangle\}$ belongs to ℓ^2. Therefore, one has, again by linearity and by (1.1),

$$\langle T, x\rangle = \sum_{h=1}^{+\infty}(x|e_h)\langle T, e_h\rangle, \quad \forall x \in H.$$

Let now y be the vector of H defined by

$$y = \sum_{h=1}^{+\infty}\langle T, e_h\rangle e_h.$$

Then, since $\langle T, e_h\rangle = (y|e_h)$, one has

$$\langle T, x\rangle = \sum_{h=1}^{+\infty}(x|e_h)(y|e_h), \quad \forall x \in H,$$

and the right hand side is nothing but the scalar product in ℓ^2 of $\mathcal{F}(x)$ and $\mathcal{F}(y)$. Since $\mathcal{F}$ is an isometry, we then have

$$\langle T, x\rangle = (y|x), \quad \forall x \in H,$$

as desired. Uniqueness follows from the fact that $(y|x) = (z|x)$ for every x in H implies $y = z$ (just take $x = y - z$). □

Corollary 1.6. *The map $T \mapsto y$ is a bijective linear isometry between H' and H.*

Proof. Since $\langle T + S, x\rangle = \langle T, x\rangle + \langle S, x\rangle$, and $\langle \lambda T, x\rangle = \lambda\langle T, x\rangle$, it is clear that the map $T \mapsto y$ is linear. In order to prove that it is an isometry, we have

$$|\langle T, x\rangle| = |(y|x)| \leq \|y\|\|x\|,$$

which implies $\|T\| \leq \|y\|$. Furthermore

$$\|y\|^2 = (y|y) = \langle T, y\rangle \leq \|T\|\|y\|,$$

so that $\|y\| \leq \|T\|$. The map is clearly injective, and it is surjective since the application $x \mapsto (y|x)$ is linear and continuous on H (by Cauchy-Schwartz inequality). □

2.2. *Bilinear forms.* An application $a : H \times H \to \mathbb{R}$ such that

$$a(\lambda x + \mu y, z) = \lambda a(x, z) + \mu a(y, z),$$

and

$$a(z, \lambda x + \mu y) = \lambda a(z, x) + \mu a(z, x),$$

for every x and y in H, and for every λ and μ in $\mathbb{R}$, is called *bilinear form*. A bilinear form is said to be *continuous* if there exists $\beta \geq 0$ such that

$$|a(x, y)| \leq \beta \|x\| \|y\|, \quad \forall x, y \in H,$$

and is said to be *coercive* if there exists $\alpha > 0$ such that

$$a(x, x) \geq \alpha \|x\|^2, \quad \forall x \in H.$$

An example of bilinear form on H is the scalar product, which is both continuous (with $\beta = 1$, thanks to the Cauchy-Schwartz inequality), and coercive (with $\alpha = 1$, by definition of the norm in H).

THEOREM 1.7. *Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear form. Then there exists a linear and continuous map $A : H \to H$ such that*

$$a(x, y) = (A(x)|y), \quad \forall x, y \in H.$$

Proof. Since a is linear in the second argument and continuous, for every fixed x in H the map $y \mapsto a(x, y)$ is linear and continuous, so that it belongs to H'. By Riesz theorem, there exists a unique vector $A(x)$ in H such that

$$a(x, y) = (A(x)|y), \quad \forall x, y \in H.$$

Since a is linear in the first argument, the map $x \mapsto A(x)$ is linear. Furthermore, by the continuity of a,

$$\|A(x)\|^2 = (A(x)|A(x)) = a(x, A(x)) \leq \beta \|x\| \|A(x)\|,$$

so that $\|A(x)\| \leq \beta \|x\|$, and the map is continuous. □

2.3. *Banach-Caccioppoli and Lax-Milgram theorems.*

THEOREM 1.8 (Banach-Caccioppoli). *Let (X, d) be a complete metric space, and let $S : X \to X$ be a contraction mapping, i.e., a continuous application such that there exists θ in $[0, 1)$ such that*

$$d(S(x), S(y)) \leq \theta\, d(x, y), \quad \forall x, y \in X.$$

Then there exists a unique $\overline{x}$ in X such that $S(\overline{x}) = \overline{x}$.

Proof. Let x_0 in X be fixed, and define $x_1 = S(x_0)$, $x_2 = S(x_1)$, and, in general, $x_n = S(x_{n-1})$. We then have, since S is a contraction mapping,

$$d(x_{n+1}, x_n) = d(S(x_n), S(x_{n-1})) \leq \theta\, d(x_n, x_{n-1}),$$

and iterating we obtain

$$d(x_{n+1}, x_n) \leq \theta^n\, d(x_1, x_0).$$

Therefore, by the triangular inequality,

$$d(x_n, x_m) \leq \sum_{h=m}^{n-1} d(x_{h+1}, x_h) \leq \sum_{h=m}^{n-1} \theta^h\, d(x_1, x_0) = \frac{\theta^m - \theta^n}{1-\theta}.$$

Since $\{\theta^h\}$ is a Cauchy sequence in $\mathbb{R}$ (being convergent to zero), it then follows that $\{x_n\}$ is a Cauchy sequence in (X, d), which is complete. Therefore, there exists $\overline{x}$ in X such that x_n converges to $\overline{x}$. Since S is continuous, on one hand $S(x_n)$ converges to $S(\overline{x})$, and on the other hand $S(x_n) = x_{n+1}$ converges to $\overline{x}$ so that $\overline{x}$ is a fixed point for S. If there exist $\overline{x}$ and $\overline{y}$ such that $S(\overline{x}) = \overline{x}$ and $S(\overline{y}) = \overline{y}$, then, since S is a contraction mapping,

$$d(\overline{x}, \overline{y}) = d(S(\overline{x}), S(\overline{y})) \leq \theta\, d(\overline{x}, \overline{y}),$$

which implies (since $\theta < 1$) $d(\overline{x}, \overline{y}) = 0$ and so $\overline{x} = \overline{y}$. □

THEOREM 1.9 (Lax-Milgram). *Let $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form, and let T be an element of H'. Then there exists a unique $\overline{x}$ in H such that*

$$a(\overline{x}, z) = \langle T, z \rangle, \quad \forall z \in H. \tag{1.2}$$

Proof. Using the Riesz theorem and Theorem 1.7, solving the equation (1.2) is equivalent to find $\overline{x}$ such that

$$a(\overline{x}, z) = (A(\overline{x})|z) = (y|z) = \langle T, z \rangle, \quad \forall z \in H,$$

i.e., to solve the equation $A(\overline{x}) = y$. Given $\lambda > 0$, this equation is equivalent to $\overline{x} = \overline{x} - \lambda A(\overline{x}) + \lambda y$, which is a fixed point problem for the function $S(x) = x - \lambda A(x) - \lambda y$. Since, being A linear, one has

$$S(x_1) - S(x_2) = x_1 - x_2 - \lambda A(x_1) + \lambda A(x_2) = x_1 - x_2 - \lambda A(x_1 - x_2),$$

in order to prove that S is a contraction mapping, it is enough to prove that there exists $\lambda > 0$ such that

$$\|x - \lambda A(x)\| \leq \theta \|x\|,$$

for some $\theta < 1$ and for every x in H. We have

$$\|x - \lambda A(x)\|^2 = \|x\|^2 + \lambda^2 \|A(x)\|^2 - 2\lambda (A(x)|x).$$

Recalling Theorem 1.7 and the definition of A, we have

$$\|A(x)\|^2 \leq \beta^2 \|x\|^2, \quad (A(x)|x) = a(x,x) \geq \alpha \|x\|^2,$$

so that

$$\|x - \lambda A(x)\|^2 \leq (1 + \lambda^2 \beta^2 - 2\lambda\alpha)\|x\|^2.$$

If $0 < \lambda < \frac{2\alpha}{\beta^2}$, we have $\theta^2 = 1 + \lambda^2\beta^2 - 2\lambda\alpha < 1$, so that S is a contraction mapping. $\square$

3. Sobolev spaces

The Banach spaces where we will look for solutions are space of functions in Lebesgue spaces "with derivatives in Lebesgue spaces" (whatever this means).

3.1. *Definition of Sobolev spaces.* Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 1$, and let u be a function in $L^1(\Omega)$. We say that u has a *weak* (or *distributional*) *derivative* in the direction x_i if there exists a function v in $L^1(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega v\,\varphi, \quad \forall \varphi \in C_0^1(\Omega).$$

In this case we define the weak derivative $\frac{\partial u}{\partial x_i}$ as the function v. If u has weak derivatives in every direction, we define its (weak, or distributional) gradient as the vector

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

If $p \geq 1$, we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N \right\}.$$

The Sobolev space $W^{1,p}(\Omega)$ becomes a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{(L^p(\Omega))^N},$$

and $W^{1,2}(\Omega)$ is a Hilbert space under the scalar product

$$(u|v)_{W^{1,2}(\Omega)} = \int_\Omega u\,v + \int_\Omega \nabla u \cdot \nabla v.$$

For historical reasons the space $W^{1,2}(\Omega)$ is usually denoted by $H^1(\Omega)$: we will use this notation from now on.

Since we will be dealing with elliptic problems with zero boundary conditions, we need to define functions which somehow are "zero" on the boundary of Ω. Since $\partial\Omega$ has zero Lebesgue measure, and functions in $W^{1,p}(\Omega)$ are only defined up to almost everywhere equivalence, there is no "direct" way of defining the boundary value a function u in some Sobolev space. We then give the following definition.

DEFINITION 1.10. We define $W_0^{1,p}(\Omega)$ as the closure of $C_0^1(\Omega)$ in the norm of $W^{1,p}(\Omega)$. If $p=2$, we will denote $W_0^{1,2}(\Omega)$ by $H_0^1(\Omega)$, which is a Hilbert space.

From now on we will mainly deal with $W_0^{1,p}(\Omega)$.

3.2. *Properties of Sobolev spaces.* Since a function in $W_0^{1,p}(\Omega)$ is "zero at the boundary" it is possible to control the norm of u in $L^p(\Omega)$ with the norm of its gradient in the same space. This is known as Poincaré inequality.

THEOREM 1.11 (Poincaré inequality). *Let $p \geq 1$; then there exists a constant C, only depending on Ω, N and p, such that*

$$\|u\|_{L^p(\Omega)} \leq C \, \|\nabla u\|_{(L^p(\Omega))^N}, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.3}$$

Proof. We only give an idea of the proof in dimension 1. Let u belong to $C_0^1((0,1))$. Then

$$u(x) = u(0) + \int_0^x u'(t)\,dt = \int_0^x u'(t)\,dt, \quad \forall x \in (0,1).$$

Thus, by Hölder inequality

$$|u(x)|^p = \left|\int_0^x u'(t)\,dt\right|^p \leq x^{\frac{p}{p'}} \int_0^x |u'(t)|^p \leq \int_0^1 |u'(t)|^p.$$

Integrating this inequality yields the result for $C_0^1((0,1))$ functions. The result for functions in $W_0^{1,p}(\Omega)$ then follows by a density argument. □

As a consequence of Poincaré inequality, we can define on $W_0^{1,p}(\Omega)$ the equivalent norm built after the norm of ∇u in $(L^p(\Omega))^N$. From now on, we will have

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{(L^p(\Omega))^N}.$$

Even though functions in $W_0^{1,p}(\Omega)$ should only belong to $L^p(\Omega)$, the assumptions made on the gradient allow to improve the summability of functions belonging to Sobolev spaces. This is what is stated in the following "embedding" theorem.

THEOREM 1.12. *Let $1 \leq p < N$, and let $p^* = \frac{Np}{N-p}$ (p^* is called the Sobolev embedding exponent). Then there exists a constant $\mathcal{S}_p$ (depending only on N and p) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq \mathcal{S}_p \, \|u\|_{W_0^{1,p}(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.4}$$

REMARK 1.13. The fact that p^* is the correct exponent can be easily recovered by a scaling argument. Indeed, if u belongs to $W_0^{1,p}(\mathbb{R}^N)$, then $u(\lambda x)$ belongs to the same space. But then

$$\int_{\mathbb{R}^N} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^N} \int_{\mathbb{R}^N} |u(y)|^q \, dy,$$

and

$$\int_{\mathbb{R}^N} |\nabla u(\lambda x)|^p \, dx = \frac{1}{\lambda^{N-p}} \int_{\mathbb{R}^N} |\nabla u(y)|^p \, dy.$$

Therefore, if (1.4) holds for some constant C (independent on λ) and some exponent q, one should have

$$\frac{N}{q} = \frac{N-p}{p},$$

which implies $q = \frac{Np}{N-p} = p^*$.

By (1.4), the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ is continuous. To obtain compactness, we cannot consider exponents up to p^*.

THEOREM 1.14. *Let* $1 \leq p < N$, *and let* $1 \leq q < p^*$. *Then the embedding of* $W_0^{1,p}(\Omega)$ *into* $L^q(\Omega)$ *is compact.*

REMARK 1.15. The fact that the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is not compact is at the basis for several nonexistence results for equations like $-\Delta u = u^q$ if q is "too large". But this is another story...

An important role will be played by the dual of a Sobolev space. We have the following representation theorem.

THEOREM 1.16. *Let* $p > 1$, *and let* T *be an element of* $(W_0^{1,p}(\Omega))'$. *Then there exists* F *in* $(L^{p'}(\Omega))^N$ *such that*

$$\langle T, u \rangle = \int_\Omega F \cdot \nabla u, \quad \forall u \in W_0^{1,p}(\Omega).$$

The dual of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$, while the dual of $H_0^1(\Omega)$ is $H^{-1}(\Omega)$.

REMARK 1.17. The space $H_0^1(\Omega)$ is a Hilbert space. Therefore, by Theorem 1.5, it is isometrically equivalent to its dual $H^{-1}(\Omega)$. Furthermore, by Poincaré inequality, $H_0^1(\Omega)$ is embedded into $L^2(\Omega)$, which is itself a Hilbert space. Since the embedding is continuous and dense, we also have that the the dual of $L^2(\Omega)$ (which is $L^2(\Omega)$) is embedded into $H^{-1}(\Omega)$. We therefore have

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset (H_0^1(\Omega))' = H^{-1}(\Omega).$$

If we identify both $L^2(\Omega)$ and its dual, **and** $H_0^1(\Omega)$ and its dual, we obtain a contradiction (since $H_0^1(\Omega)$ and $L^2(\Omega)$ are different spaces). Therefore, we have to choose which identification to make: which will be that $L^2(\Omega)$ is equivalent to its dual.

REMARK 1.18. Since, by Sobolev embedding, $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, we have by duality that $(L^{p^*}(\Omega))'$ is continuously embedded in $W^{-1,p'}(\Omega)$. If we define

$$p_* = (p^*)' = \frac{Np}{Np - N + p},$$

we then have

$$L^{p_*}(\Omega) \subset W^{-1,p'}(\Omega).$$

If $p = 2$, we have $2_* = \frac{2N}{N+2}$, and the embedding of $L^{2_*}(\Omega)$ into $H^{-1}(\Omega)$.

The final result on Sobolev spaces will be about composition with regular functions.

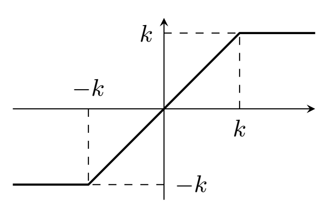
THEOREM 1.19. *Let $G : \mathbb{R} \to \mathbb{R}$ be a lipschitz continuous functions such that $G(0) = 0$. If u belongs to $W_0^{1,p}(\Omega)$, then $G(u)$ belongs to $W_0^{1,p}(\Omega)$ as well, and*

$$\nabla G(u) = G'(u)\,\nabla u, \quad \text{almost everywhere in } \Omega. \tag{1.5}$$

REMARK 1.20. Recall that a lipschitz continuous function is only almost everywhere differentiable, so that the right-hand side of (1.5) may not be defined. We have however two possible cases: if k is a value such that $G'(k)$ does not exist, either the set $\{u = k\}$ has zero measure (and so, since identity (1.5) only holds almost everywhere, this value does not give any problems), or the set $\{u = k\}$ has positive measure. In this latter case, however, we have both $\nabla u = 0$ and $\nabla G(u) = 0$ almost everywhere, so that (1.5) still holds.

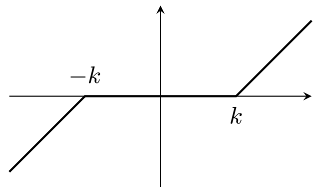
Let $k > 0$; in what follows, we will often use composition of functions in Sobolev spaces with

$$T_k(s) = \max(-k, \min(s, k)), \tag{1.6}$$



and

$$G_k(s) = s - T_k(s) = (|s| - k)_+ \mathrm{sgn}(s). \tag{1.7}$$



By Theorem 1.19, we have

$$\nabla T_k(u) = \nabla u \, \chi_{\{|u| \leq k\}}, \quad \nabla G_k(u) = \nabla u \, \chi_{\{|u| \geq k\}},$$

almost everywhere in Ω.

4. Weak solutions for elliptic equations

We have now all the tools needed to deal with elliptic equations.

4.1. *Definition of weak solutions.* Let $A : \Omega \to \mathbb{R}^{N^2}$ be a matrix-valued measurable function such that there exist $0 < \alpha \leq \beta$ such that

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |A(x)| \leq \beta, \tag{1.8}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. We will consider the following uniformly elliptic equation with Dirichlet boundary conditions

$$\begin{cases} -\text{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where f is a function defined on Ω which satisfies suitable assumptions. If the matrix A is the identity matrix, problem (1.9) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., the Dirichlet problem for the laplacian operator.

4.2. *Classical solutions and weak solutions.* Suppose that the matrix A and the functions u and f are sufficiently smooth so that one can "classically" compute $-\text{div}(A(x)\nabla u)$. If φ is a function in $C_0^1(\Omega)$, we can then multiply the equation in (1.9) by φ and integrate on Ω. Since

$$-\text{div}(A(x)\nabla u)\,\varphi = -\text{div}(A(x)\nabla u\,\varphi) + A(x)\nabla u \cdot \nabla\varphi,$$

we get

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi - \int_\Omega \text{div}(A(x)\nabla u\,\varphi) = \int_\Omega f\,\varphi.$$

By Gauss-Green formula, we have (if ν is the exterior normal to Ω)

$$\int_\Omega \text{div}(A(x)\nabla u\,\varphi) = \int_{\partial\Omega} A(x)\nabla u \cdot \nu\,\varphi = 0,$$

since φ has compact support in Ω. Therefore, if u is a classical solution of (1.9), we have

$$\int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f\, v, \quad \forall v \in C_0^1(\Omega).$$

We now remark that there is no need for A, u, φ and f to be smooth in order for the above identity to be well defined. It is indeed enough that A is a bounded matrix, that u and φ belong to $H_0^1(\Omega)$, and that f is in $L^2(\Omega)$ (or in $L^{2_*}(\Omega)$, thanks to Sobolev embedding, see Remark 1.18).

We therefore give the following definition.

DEFINITION 1.21. Let f be a function in $L^{2_*}(\Omega)$. A function u in $H_0^1(\Omega)$ is a *weak solution* of (1.9) if

$$\int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f\, v, \quad \forall v \in H_0^1(\Omega). \tag{1.10}$$

If u is a weak solution of (1.9), and u is sufficiently smooth in order to perform the same calculations as above "going backwards", then it can be proved that u is a "classical" solution of (1.9). The study of the assumptions on f and A such that a weak solution is also a classical solution goes beyond the purpose of this text (also because we are interested in "bad" data!).

4.3. *Existence of solutions (using Lax-Milgram).*

THEOREM 1.22. *Let f be a function in $L^{2_*}(\Omega)$. Then there exists a unique solution u of (1.9) in the sense of (1.10).*

Proof. We will use Lax-Milgram theorem. Indeed, if we define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega A(x)\nabla u \cdot \nabla v,$$

and the linear and continuos (thanks to Sobolev embedding) functional $T : H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle T, v\rangle = \int_\Omega f\, v,$$

solving problem (1.9) in the sense of (1.10) amounts to finding u in $H_0^1(\Omega)$ such that

$$a(u, v) = \langle T, v\rangle, \quad \forall v \in H_0^1(\Omega),$$

which is exactly the result given by Lax-Milgram theorem. In order to apply the theorem, we have to check that a is continuous and coercive (the fact that it is bilinear being evident). We have, by (1.8), and by Hölder inequality,

$$|a(u,v)| \leq \int_\Omega |A(x)||\nabla u||\nabla v| \leq \beta \, \|u\|_{H_0^1(\Omega)} \, \|v\|_{H_0^1(\Omega)} ,$$

so that a is continuous. Furthermore, again by (1.8), we have

$$a(u,u) = \int_\Omega A(x)\nabla u \cdot \nabla u \geq \alpha \int_\Omega |\nabla u|^2 = \alpha \, \|u\|^2_{H_0^1(\Omega)} ,$$

so that a is also coercive. □

4.4. *Existence of solutions (using minimization).* If the matrix A satisfies (1.8) and is symmetrical, existence and uniqueness of solutions for (1.9) can be proved using minimization of a suitable functional.

THEOREM 1.23. *Let f be a function in $L^{2_*}(\Omega)$, and let $J : H_0^1(\Omega) \to \mathbb{R}$ be defined by*

$$J(v) = \frac{1}{2}\int_\Omega A(x)\nabla v \cdot \nabla v - \int_\Omega f\, v, \quad \forall v \in H_0^1(\Omega).$$

Then J has a unique minimum u in $H_0^1(\Omega)$, which is the solution of (1.9) in the sense of (1.10).

Proof. We begin by proving that J is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$, so that a minimum will exist by Theorem 1.4. Recalling (1.8) and using Hölder and Sobolev inequalities, we have

$$\begin{aligned} J(v) &\geq \frac{\alpha}{2}\int_\Omega |\nabla v|^2 - \|f\|_{L^{2_*}(\Omega)} \, \|u\|_{L^{2^*}(\Omega)} \\ &\geq \frac{\alpha}{2} \, \|u\|^2_{H_0^1(\Omega)} - \mathcal{S}_2 \, \|f\|_{L^{2_*}(\Omega)} \, \|u\|_{H_0^1(\Omega)} , \end{aligned}$$

and the right hand side diverges as the norm of u in $H_0^1(\Omega)$ diverges, so that J is coercive. Let now $\{v_n\}$ be a sequence of functions which is weakly convergent to some v in $H_0^1(\Omega)$. Since f belongs to $L^{2_*}(\Omega)$, and v_n converges weakly to v in $L^{2^*}(\Omega)$, we have

$$\lim_{n\to+\infty} \int_\Omega f\, v_n = \int_\Omega f\, v,$$

so that the weak lower semicontinuity of J is equivalent to the weak lower semicontinuity of

$$K(v) = \int_\Omega A(x)\nabla v \cdot \nabla v.$$

By (1.8) we have

$$\int_\Omega A(x)\,\nabla(v - v_n)\cdot\nabla(v - v_n) \geq 0,$$

which, together with the symmetry of A, implies

$$2\int_\Omega A(x)\nabla v\cdot\nabla v_n - \int_\Omega A(x)\nabla v\cdot\nabla v \leq \int_\Omega A(x)\nabla v_n\cdot\nabla v_n. \tag{1.11}$$

Since ∇v_n converges weakly to ∇v in $(L^2(\Omega))^N$, and since $A(x)\nabla v$ is fixed in the same space, we have

$$\lim_{n\to+\infty}\int_\Omega A(x)\nabla v\cdot\nabla v_n = \int_\Omega A(x)\nabla v\cdot\nabla v,$$

so that taking the inferior limit in both sides of (1.11) implies

$$\int_\Omega A(x)\nabla v\cdot\nabla v \leq \liminf_{n\to+\infty}\int_\Omega A(x)\nabla v_n\cdot\nabla v_n,$$

which means that K is weakly lower semicontinuous on $H_0^1(\Omega)$, as desired.

Let now u be a minimum of J on $H_0^1(\Omega)$. We are going to prove that it is unique. Indeed, if u and v are both minima of J, one has

$$J(u) \leq J\Big(\frac{u+v}{2}\Big), \quad J(v) \leq J\Big(\frac{u+v}{2}\Big),$$

that is,

$$J(u) + J(v) \leq 2J\Big(\frac{u+v}{2}\Big),$$

which becomes (after cancelling equal terms and multiplying by 4)

$$2\int_\Omega A(x)\nabla u\cdot\nabla u + 2\int_\Omega A(x)\nabla u\cdot\nabla u = \int_\Omega A(x)\nabla(u+v)\cdot\nabla(u+v).$$

Using the fact that A is symmetric, expanding the right hand side, and cancelling equal terms, we arrive at

$$\int_\Omega A(x)\nabla u\cdot\nabla u - 2\int_\Omega A(x)\nabla u\cdot\nabla v + \int_\Omega A(x)\nabla v\cdot\nabla v \leq 0,$$

which can be rewritten as

$$\int_\Omega A(x)\nabla(u-v)\cdot\nabla(u-v) \leq 0.$$

Using (1.8) we therefore have

$$\alpha\,\|u-v\|^2_{H_0^1(\Omega)} \leq 0,$$

which implies $u = v$, as desired.

We are now going to prove that the minimum u is a solution of (1.9) in the sense of (1.10). Given v in $H_0^1(\Omega)$ and t in $\mathbb{R}$, we have $J(u) \leq J(u + tv)$, that is

$$\frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega fu \leq \frac{1}{2}\int_\Omega A(x)\nabla(u+tv)\cdot\nabla(u+tv) - \int_\Omega f(u+tv).$$

Expanding the right hand side, cancelling equal terms, and using the fact that A is symmetric, we obtain

$$t\int_\Omega A(x)\nabla u\cdot\nabla v + \frac{t^2}{2}\int_\Omega A(x)\nabla v\cdot\nabla v - t\int_\Omega f\,v \geq 0.$$

If $t > 0$, dividing by t and then letting t tend to zero implies

$$\int_\Omega A(x)\nabla u\cdot\nabla v - \int_\Omega f\,v \geq 0,$$

while if $t < 0$, dividing by t and then letting t tend to zero implies the reverse inequality. It then follows that

$$\int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f\,v, \quad \forall v \in H_0^1(\Omega),$$

and so u solves (1.9) (in the sense of (1.10)). In order to prove that such a solution is unique, we are going to prove that if u solves (1.9), then u is a minimum of J. Indeed, choosing $u - v$ as test function in (1.10), we have

$$\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f(u-v).$$

This implies

$$J(u) + \frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = J(v) - \frac{1}{2}\int_\Omega A(x)\nabla v\cdot\nabla v,$$

which implies $J(u) \leq J(v)$ since

$$\frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v + \frac{1}{2}\int_\Omega A(x)\nabla v\cdot\nabla v$$

is nonnegative being equal to

$$\frac{1}{2}\int_\Omega A(x)\nabla(u-v)\cdot\nabla(u-v),$$

which is nonnegative by (1.8). $\square$

CHAPTER 2

Regularity results

Thanks to the results of the previous section, we have existence of solutions for data f in $L^{2_*}(\Omega)$. The solution u then belongs to $H^1_0(\Omega)$ and (thanks to Sobolev embedding) to $L^{2^*}(\Omega)$. One then wonders whether an increase on the regularity of f will yield more regular solutions.

1. Examples

We are going to study a model case, in which the solution of (1.9) can be explictly calculated. This example will give us a hint on what happens in the general case.

EXAMPLE 2.1. Let $\Omega = B_{\frac{1}{2}}(0)$, let $N \geq 3$, let $\alpha < N$ and define

$$f(x) = \frac{1}{|x|^\alpha\,(-\log(|x|))}.$$

It is well known that f belongs to $L^p(\Omega)$, with $p = \frac{N}{\alpha}$. We are going to study the regularity of the solution u of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

taking advantage of the fact that the solution will be radially symmetric. Recalling the formula for the laplacian in radial coordinates, we have

$$-\frac{1}{\rho^{N-1}}(\rho^{N-1}u'(\rho))' = \frac{1}{\rho^\alpha\,(-\log(\rho))}.$$

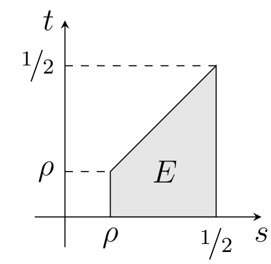
Multiplying by ρ^{N-1} and integrating between 0 and ρ, we obtain

$$\rho^{N-1}\,u'(\rho) = \int_0^\rho \frac{t^{N-1-\alpha}}{\log(t)}\,dt.$$

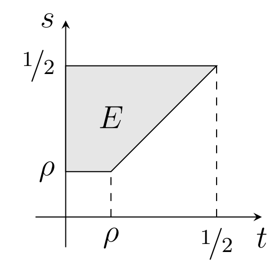
Dividing by ρ^{N-1} and integrating between $\frac{1}{2}$ and ρ we then get (recalling that $u(\frac{1}{2}) = 0$)

$$u(\rho) = -\int_\rho^{\frac{1}{2}} \frac{1}{s^{N-1}}\left(\int_0^s \frac{t^{N-1-\alpha}}{\log(t)}\,dt\right)ds.$$

We are integrating on the set $E = \{(s,t) \in \mathbb{R}^2 : \rho \le s \le \frac{1}{2},\ 0 \le t \le s\}$,



which, after exchanging t with s, becomes $E = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le \frac{1}{2},\ \max(\rho,t) \le s \le \frac{1}{2}\}$,



Exchanging the integration order, we then have

$$\begin{aligned} u(\rho) &= -\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left(\int_{\max(\rho,t)}^{\frac{1}{2}} \frac{ds}{s^{N-1}} \right) dt \\ &= \frac{1}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left[\left(\frac{1}{2}\right)^{2-N} - (\max(\rho,t))^{2-N} \right] dt \\ &= \frac{2^{N-2}}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} dt - \frac{1}{N-2} \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)} dt. \end{aligned}$$

Since $\alpha < N$, the first integral is bounded, so that it is enough to study the behaviour near zero of the function

$$\begin{aligned} v(\rho) &= \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)} dt \\ &= \rho^{2-N} \int_0^{\rho} \frac{t^{N-1-\alpha}}{\log(t)} dt + \int_\rho^{\frac{1}{2}} \frac{t^{1-\alpha}}{\log(t)} dt \\ &= \rho^{2-N} w(\rho) + z(\rho). \end{aligned}$$

It is easy to see (using the de l'Hopital rule), that if $\alpha \ne 2$

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero, so that, if $\alpha \neq 2$,

$$u(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero. This implies that u belongs to $L^\infty(\Omega)$ if $\alpha < 2$, while it is in $L^m(\Omega)$, with $m = \frac{N}{\alpha-2}$, if $2 < \alpha < N$. Recalling that f belongs to $L^p(\Omega)$ with $p = \frac{N}{\alpha}$, we therefore have that u belongs to $L^\infty(\Omega)$ if f belongs to $L^p(\Omega)$, and $p > \frac{N}{2}$, while it is in $L^m(\Omega)$, with $m = \frac{Np}{N-2p}$, if f belongs to $L^p(\Omega)$, with $1 < p < \frac{N}{2}$.

If $\alpha = 2$, then

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \log(-\log(\rho)),$$

so that u is in every $L^m(\Omega)$, but not in $L^\infty(\Omega)$, if f belongs to $L^p(\Omega)$ with $p = \frac{N}{2}$. In this case (which we will not study in the following), it can be proved that $\mathrm{e}^{|u|}$ belongs to $L^1(\Omega)$.

Observe that if $\alpha = \frac{N+2}{2}$, so that f belongs to $L^{2_*}(\Omega)$, we get that u belongs to $L^{2^*}(\Omega)$, which is exactly the results we already knew by Sobolev embedding. Also remark that the above example gives informations also if f does not belong to $L^{2_*}(\Omega)$ (i.e., if $\frac{N+2}{2} < \alpha < N$), an assumption under which we do not have any existence results (yet!).

If we want to take $\alpha = N$, we need to change the definition of f. We fix $\beta > 1$ and define

$$f(x) = \frac{1}{|x|^N\,(-\log(|x|))^\beta},$$

which is a function belonging to $L^1(\Omega)$. Performing the same calculations as above, we obtain

$$u(\rho) = \frac{1}{\beta - 1}\int_\rho^{\frac{1}{2}} \frac{dt}{t^{N-1}\,(-\log(t))^{\beta-1}},$$

so that

$$u(\rho) \approx \frac{1}{\rho^{N-2}\,(-\log(\rho))^{\beta-1}},$$

as ρ tends to zero. Observe that in this case f belongs to $L^1(\Omega)$ for every $\beta > 1$, but u belongs to $L^m(\Omega)$, with $m = \frac{N\cdot 1}{N-2\cdot 1} = \frac{N}{N-2}$ if and only if $\beta > 2 - \frac{2}{N}$. If $1 < \beta \leq 2 - \frac{2}{N}$, the solution u belongs "only" to $L^m(\Omega)$, for every $m < \frac{N}{N-2}$.

We leave to the interested reader the study of the case $N = 2$.

2. Stampacchia's theorems

The regularity results we are going to prove now show that the previous example is not just an example. We begin with a real analysis lemma.

LEMMA 2.2 (Stampacchia). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leq \frac{M\,\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0, \tag{2.12}$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where

$$d^\gamma = M\,\psi(0)^{\delta-1}\,2^{\frac{\delta\gamma}{\delta-1}}.$$

Proof. Let n in $\mathbb{N}$ and define $d_n = d(1-2^{-n})$. We claim that

$$\psi(d_n) \leq \psi(0)\,2^{-\frac{n\gamma}{\delta-1}}. \tag{2.13}$$

Indeed, (2.13) is clearly true if $n = 0$; if we suppose that it is true for some n, then, by (2.12),

$$\psi(d_{n+1}) \leq \frac{M\,\psi(d_n)^\delta}{(d_{n+1}-d_n)^\gamma} \leq M\,\psi(0)^\delta\,2^{-\frac{n\gamma\delta}{\delta-1}}\,2^{(n+1)\gamma}\,d^{-\gamma} = \psi(0)\,2^{-\frac{(n+1)\gamma}{\delta-1}},$$

which is (2.13) written for $n+1$. Since (2.13) holds for every n, and since ψ is non increasing, we have

$$0 \leq \psi(d) \leq \liminf_{n\to+\infty} \psi(d_n) \leq \lim_{n\to+\infty} \psi(0)^{\delta-1}\,2^{-\frac{n\gamma}{\delta-1}} = 0,$$

as desired. □

The first result (due to Guido Stampacchia, see [**8**]), deals with bounded solutions for (1.9).

THEOREM 2.3 (Stampacchia). *Let f belong to $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^\infty(\Omega)$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^p(\Omega)}\,. \tag{2.14}$$

Proof. Let $k > 0$ and choose $v = G_k(u)$ as test function in (1.9) ($G_k(s)$ has been defined in (1.7)). Defining $A_k = \{x \in \Omega : |u(x)| \geq k\}$ one then has, since $\nabla v = \nabla u\,\chi_{A_k}$, and using (1.8)

$$\alpha\int_{A_k} |\nabla G_k(u)|^2 \leq \int_\Omega A(x)\nabla u\cdot\nabla u\,\chi_{A_k} = \int_\Omega f\,G_k(u) = \int_{A_k} f\,G_k(u).$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right hand side), one has

$$\frac{\alpha}{\mathcal{S}_2^2}\left(\int_{A_k}|G_k(u)|^{2^*}\right)^{\frac{2}{2^*}} \le \left(\int_{A_k}|f|^{2_*}\right)^{\frac{1}{2_*}}\left(\int_{A_k}|G_k(u)|^{2^*}\right)^{\frac{1}{2^*}}.$$

Simplifying equal terms, we thus have

$$\int_{A_k}|G_k(u)|^{2^*} \le \left(\frac{\mathcal{S}_2^2}{\alpha}\right)^{2^*}\left(\int_{A_k}|f|^{2_*}\right)^{\frac{2^*}{2_*}}.$$

Recalling that f belongs to $L^p(\Omega)$, and that $p > 2_*$ since $p > \frac{N}{2}$, we have (again by Hölder inequality)

$$\int_{A_k}|G_k(u)|^{2^*} \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*} m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}}.$$

We now take $h > k$, so that $A_h \subseteq A_k$, and $G_k(u) \ge h-k$ on A_h. Thus,

$$(h-k)^{2^*}\,m(A_h) \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*} m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}},$$

which implies

$$m(A_h) \le \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*}\frac{m(A_k)^{\frac{2^*}{2_*}-\frac{2^*}{p}}}{(h-k)^{2^*}}.$$

We define now $\psi(k) = m(A_k)$, so that

$$\psi(h) \le \frac{M\,\psi(k)^\delta}{(h-k)^\gamma},$$

where

$$M = \left(\frac{\mathcal{S}_2^2\,\|f\|_{L^p(\Omega)}}{\alpha}\right)^{2^*}, \quad \delta = \frac{2^*}{2_*} - \frac{2^*}{p}, \quad \gamma = 2^*.$$

The assumption $p > \frac{N}{2}$ implies $\delta > 1$, so that applying Lemma 2.2, we have that $\psi(d) = 0$, where

$$d^{2^*} = C(\Omega, N, p)\,M.$$

Since $m(A_d) = 0$, we have $|u| \le d$ almost everywhere, which implies

$$\|u\|_{L^\infty(\Omega)} \le d = C(N,\Omega,p,\alpha)\ \|f\|_{L^p(\Omega)},$$

as desired. □

Remark 2.4. Observe that, in order to prove the previous theorem, we did not use two of the properties of the equation: that the matrix A is bounded from above (we only used its ellipticity) and, above all, the fact that the equation was *linear*: in other words, the proof above also holds for every uniformly elliptic operator.

The second results deals with the case of unbounded solutions.

Theorem 2.5 (Stampacchia). *Let f belong to $L^p(\Omega)$, with $2_* \leq p < \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^m(\Omega)$, with $m = p^{**} = \frac{Np}{N-2p}$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^{p^{**}}(\Omega)} \leq C \, \|f\|_{L^p(\Omega)} \,. \tag{2.15}$$

Proof. We begin by observing that if $p = 2_*$, then $p^{**} = 2^*$, so that the result is true in this limit case by the Sobolev embedding. Therefore, we only have to deal with the case $p > 2_*$.

The original proof of Stampacchia used a linear interpolation theorem; i.e., it is typical of a linear framework. We are going to give another proof, following [**3**], which makes use of a technique that can be applied also in a nonlinear context.

Let $k > 0$ be fixed, let $\gamma > 1$ and choose $v = |T_k(u)|^{2\gamma-2}\, T_k(u)$ as test function in (1.9) ($T_k(s)$ has been defined in (1.6)). We obtain

$$(2\gamma - 1)\int_\Omega A(x)\nabla u \cdot \nabla T_k(u)\, |T_k(u)|^{2\gamma-2} = \int_\Omega f\, |T_k(u)|^{2\gamma-2}\, T_k(u).$$

Using (1.8), and observing that $\nabla u = \nabla T_k(u)$ where $\nabla T_k(u) \neq 0$, we then have

$$\alpha\,(2\gamma - 1)\int_\Omega |\nabla T_k(u)|^2\, |T_k(u)|^{2\gamma-2} \leq \int_\Omega |f|\, |T_k(u)|^{2\gamma-1}.$$

Since $|\nabla T_k(u)|^2\, |T_k(u)|^{2\gamma-2} = \frac{1}{\gamma^2}\, |\nabla |T_k(u)|^\gamma|^2$, we have

$$\frac{\alpha\,(2\gamma-1)}{\gamma^2}\int_\Omega |\nabla |T_k(u)|^\gamma|^2 \leq \int_\Omega |f|\, |T_k(u)|^{2\gamma-1}.$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right one), we obtain

$$\frac{\alpha\,(2\gamma-1)}{\mathcal{S}_2^2\gamma^2}\left(\int_\Omega |T_k(u)|^{\gamma 2^*}\right)^{\frac{2}{2^*}} \leq \|f\|_{L^p(\Omega)}\left(\int_\Omega |T_k(u)|^{(2\gamma-1)p'}\right)^{\frac{1}{p'}}.$$

We now choose γ so that $\gamma 2^* = (2\gamma - 1)p'$, that is $\gamma = \frac{p^{**}}{2^*}$ (as it is easily seen). With this choice, $\gamma > 1$ if and only if $p > 2_*$ (which is true). Since $p < \frac{N}{2}$, we also have $\frac{2}{2^*} > \frac{1}{p'}$, and so

$$\left(\int_\Omega |T_k(u)|^{p^{**}} \right)^{\frac{2}{2^*} - \frac{1}{p'}} \le C(N, \Omega, p, \alpha)\ \|f\|_{L^p(\Omega)}.$$

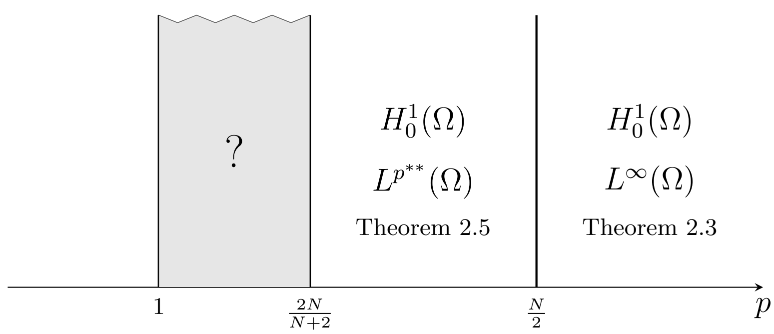
Observing that $\frac{2}{2^*} - \frac{1}{p'} = \frac{1}{p^{**}}$, we have therefore proved that

$$\|T_k(u)\|_{L^{p^{**}}(\Omega)} \le C(N, \Omega, p, \alpha)\ \|f\|_{L^p(\Omega)}, \quad \forall k > 0.$$

Letting k tend to infinity, and using Fatou lemma (or the monotone convergence theorem), we obtain the result. □

REMARK 2.6. The results of theorems 2.3 and 2.5 are somehow "natural" if we make a mistake... Indeed, let u be the solution of $-\Delta u = f$, with f in $L^p(\Omega)$. Then, if we read the equation, we have that u has two derivatives in $L^p(\Omega)$, so that it belongs to $W_0^{2,p}(\Omega)$. By Sobolev embedding, u then belongs to $W_0^{1,p^*}(\Omega)$ and, again by Sobolev embedding, to $L^{p^{**}}(\Omega)$ (or to $L^\infty(\Omega)$ if $p > \frac{N}{2}$). The "mistake" here is to deduce from the fact that the sum of (some) derivatives of u belongs to $L^p(\Omega)$, the fact that all derivatives are in the same space. Surprisingly, it turns out that, in the case of the laplacian, the fact that $-\Delta u$ belongs to $L^p(\Omega)$ actually implies that u is in $W_0^{2,p}(\Omega)$ (this is the so-called Calderun-Zygmund theory), so that the "mistake" is not an actual one...

Summarizing the results of this chapter, we have the following picture.



We will deal with the "?" part in the forthcoming chapter (actually, in all the forthcoming chapters).

CHAPTER 3

Existence via duality for measure data

We are now going to deal with existence results for data which do not belong to $L^{2_*}(\Omega)$ (i.e., they are not in $H^{-1}(\Omega)$), so that neither Lax-Milgram theorem nor minimization techniques can be applied. Before going on, we need some definitions.

1. Measures

We recall that a *nonnegative measure* on Ω is a set function $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ defined on the σ-algebra $\mathcal{B}(\Omega)$ of Borel sets of Ω (i.e., the smallest σ-algebra containing the open sets) such that $\mu(\emptyset) = 0$ and such that

$$\mu\Big(\bigcup_{n=1}^{+\infty} E_n\Big) = \sum_{n=1}^{+\infty} \mu(E_n),$$

for every sequence $\{E_n\}$ of disjoint sets in $\mathcal{B}(\Omega)$. A measure μ is said to be *regular* if for every E in $\mathcal{B}(\Omega)$ and for every $\varepsilon > 0$ there exist an open set A_ε, and a closed set C_ε, such that

$$C_\varepsilon \subseteq E \subseteq A_\varepsilon, \quad \mu(A_\varepsilon \setminus C_\varepsilon) < \varepsilon.$$

A measure μ is said to be *bounded* if $\mu(\Omega) < +\infty$. The set of nonnegative, regular, bounded measures on Ω will be denoted by $\mathcal{M}^+(\Omega)$. We define the set of *bounded Radon measures* on Ω as

$$\mathcal{M}(\Omega) = \{\mu_1 - \mu_2,\ \mu_i \in \mathcal{M}^+(\Omega)\}.$$

Given a measure μ in $\mathcal{M}(\Omega)$, there exists a unique pair (μ^+, μ^-) in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\Omega)$ such that

$$\mu = \mu^+ - \mu^-,$$

and such there exist E^+ and E^- in $\mathcal{B}(\Omega)$, *disjoint sets*, such that

$$\mu^\pm(E) = \mu(E \cap E^\pm), \quad \forall E \in \mathcal{B}(\Omega).$$

The measures μ^+ and μ^- are the *positive* and *negative* parts of the measure μ. Given a measure μ in $\mathcal{M}(\Omega)$, the measure $|\mu| = \mu^+ + \mu^-$ is said to be the *total variation* of the measure μ. If we define

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega),$$

the vector space $\mathcal{M}(\Omega)$ becomes a Banach space, which turns out to be the dual of $C_0^0(\Omega)$.

A bounded Radon measure μ is said to be *concentrated* on a Borel set E if $\mu(B) = \mu(B \cap E)$ for every Borel set B. In this case, we will write $\mu \llcorner E$. For example, we have $\mu^{\pm} = \mu \llcorner E^{\pm}$, with $E^{\pm}$ as above.

Given two Radon measures μ and ν, we say that μ is *absolutely continuous* with respect to ν if $\nu(E) = 0$ implies $\mu(E) = 0$. In this case we will write $\mu << \nu$. Two Radon measures μ and ν are said to be *orthogonal* if there exists a set E such that $\mu(E) = 0$, and $\nu = \nu \llcorner E$. In this case, we will write $\mu \perp \nu$. For example, given a Radon measure μ, we have $\mu^+ \perp \mu^-$.

THEOREM 3.1. *Let ν be a nonnegative Radon measure. Given a Radon measure μ, there exists a unique pair (μ_0, μ_1) of Radon measures such that*

$$\mu = \mu_0 + \mu_1, \quad \mu_0 << \nu, \quad \mu_1 \perp \nu.$$

Proof. Suppose that μ is nonnegative, and define

$$\mathcal{A} = \{\mu(E) : E \in \mathcal{B}(\Omega),\ \nu(E) = 0\}.$$

Let $\alpha = \sup \mathcal{A}$, and let E_n be a maximizing sequence, i.e., a sequence of Borel sets such that

$$\lim_{n \to +\infty} \mu(E_n) = \alpha, \quad \nu(E_n) = 0.$$

If we define E as the union of the E_n, clearly $\nu(E) = 0$ (since ν is σ-subadditive), and $\mu(E) = \alpha$ (since $\mu(E) \geq \mu(E_n)$ for every n). Define now

$$\mu_1 = \mu \llcorner E, \quad \mu_0 = \mu - \mu_1.$$

Clearly, $\mu_1 \perp \nu$ (since $\nu(E) = 0$, and since μ_1 is concentrated on E by definition). On the other hand, if $\nu(B) = 0$, then $\mu_0(B) = 0$; and indeed, if it were $\mu_0(B) > 0$, then

$$0 < \mu_0(B) = \mu(B) - \mu(B \cap E) = \mu(B \setminus E),$$

so that $B \cup E$ is such that $\nu(B \cup E) = 0$, and

$$\mu(B \cup E) = \mu(E) + \mu(B \setminus E) = \alpha + \mu(B \setminus E) > \alpha,$$

thus contradicting the definition of α.

As for uniqueness, if $\mu = \mu_0 + \mu_1 = \mu_0' + \mu_1'$, then $\mu_0 - \mu_0' = \mu_1' - \mu_1$. If $\nu(B) = 0$, we will have $(\mu_1 - \mu_1')(B) = 0$. Since $\mu_1 - \mu_1'$ is orthogonal with respect to ν, this implies that $(\mu_1 - \mu_1')(E) = 0$ for every Borel set E, so that $\mu_1 = \mu_1'$, hence $\mu_0 = \mu_0'$.

If the measure μ has a sign, it is enough to apply the result to μ^+ and μ^-. $\square$

Examples of bounded Radon measures are the Lebesgue measure $\mathcal{L}^N$ concentrated on a bounded set of $\mathbb{R}^N$, or the measure defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E, \end{cases}$$

which is called the *Dirac's delta* concentrated at x_0. We clearly have $\delta_{x_0} \perp \mathcal{L}^N$. Another example of Radon measure is the measure defined by

$$\mu(E) = \int_E f(x)\,dx,$$

with f a function in $L^1(\Omega)$. In this case $\mu << \mathcal{L}^N$, and

$$\mu^{\pm}(E) = \int_E f^{\pm}(x)\,dx, \quad |\mu|(E) = \int_E |f(x)|\,dx.$$

For sequences of measures, we have two notions of convergence: the weak*:

$$\int_\Omega \varphi\,d\mu_n \to \int_\Omega \varphi\,d\mu, \quad \forall \varphi \in C^0_0(\Omega),$$

and the *narrow convergence*:

$$\int_\Omega \varphi\,d\mu_n \to \int_\Omega \varphi\,d\mu, \quad \forall \varphi \in C^0_{\mathrm{b}}(\Omega).$$

For positive measures, narrow convergence is equivalent to weak* convergence and convergence of the "masses" (i.e., $\mu_n(\Omega)$ converges to $\mu(\Omega)$). If x_n is a sequence in Ω which converges to a point x_0 on $\partial\Omega$, then δ_{x_n} converges to zero for the weak* convergence (since the measure δ_{x_0} is indeed the zero measure in Ω), but not for the narrow convergence.

Before dealing with existence results for elliptic equations with measure data, we will begin with a particular case.

2. Duality solutions for L^1 data

Let f and g be two functions in $L^\infty(\Omega)$, and let u and v be the solutions of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

where A^* is the transposed matrix of A (note that A^* satisfies (1.8) with the same constants as A). Since both u and v belong to $H^1_0(\Omega)$, u

can be chosen as test function in the formulation of weak solution for v, and *vice versa*. One obtains

$$\int_\Omega u\,g = \int_\Omega A^*(x)\nabla v \cdot \nabla u = \int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f\,v.$$

In other words, one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every f and g in $L^\infty(\Omega)$, where u and v solve the corresponding problems with data f and g respectively. Clearly, both u and v belong to $L^\infty(\Omega)$ by Theorem 2.3, but we remark that the two integrals are well-defined also if f only belongs to $L^1(\Omega)$, and u only belongs to $L^1(\Omega)$ (always maintaining the assumption that g — and so v — is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (1.9) if the datum is in $L^1(\Omega)$.

Definition 3.2. Let f belong to $L^1(\Omega)$. A function u in $L^1(\Omega)$ is a *duality solution* of (1.8) with datum f if one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every g in $L^\infty(\Omega)$, where v is the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3.3 (Stampacchia). *Let f belong to $L^1(\Omega)$. Then there exists a unique duality solution of (1.8) with datum f. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.*

Proof. Let $p > \frac{N}{2}$ and define the linear functional $T : L^p(\Omega) \to \mathbb{R}$ by

$$\langle T, g\rangle = \int_\Omega f\,v.$$

By Theorem 2.3, the functional is well-defined; furthermore, since (2.14) holds, there exists $C > 0$ such that

$$|\langle T, g\rangle| \le \int_\Omega |f|\,|v| \le \|f\|_{L^1(\Omega)}\,\|v\|_{L^\infty(\Omega)} \le C\,\|f\|_{L^1(\Omega)}\,\|g\|_{L^p(\Omega)},$$

so that T is continuous on $L^p(\Omega)$. By Riesz representation Theorem for L^p spaces, there exists a unique function u_p in $L^{p'}(\Omega)$ such that

$$\langle T, g\rangle = \int_\Omega u_p\,g, \quad \forall g \in L^p(\Omega).$$

Since $L^\infty(\Omega) \subset L^p(\Omega)$, we have

$$\int_\Omega u_p\, g = \langle T, g\rangle = \int_\Omega f\, v, \quad \forall g \in L^\infty(\Omega),$$

so that u_p is a duality solution of (1.9), as desired. We claim that u_p does not depend on p; indeed, if for example $p > q > \frac{N}{2}$, we have

$$\int_\Omega u_p\, g = \int_\Omega f\, v = \int_\Omega u_q\, g, \quad \forall g \in L^\infty(\Omega),$$

so that $u_p = u_q$ in $L^1(\Omega)$ (and so they are almost everywhere the same function). Therefore, there exists a unique function u which is a duality solution of (1.9), and it belongs to $L^{p'}(\Omega)$ for every $p > \frac{N}{2}$; i.e., u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$, as desired. □

Remark that the fact that u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$ is consistent with the results of the last part of Example 2.1 (the case $\alpha = N$).

3. Duality solutions for measure data

The case of $L^1(\Omega)$ data is only a particular one, since $L^1(\Omega)$ is a subset of $\mathcal{M}(\Omega)$. However, recalling that $\mathcal{M}(\Omega)$ is the dual of $C^0(\overline{\Omega})$, the proof of Theorem 3.3 could be performed in exactly the same way if one knew that the solution of (1.9) were not only bounded, but also *continuous* on Ω if the datum is in $L^p(\Omega)$ with $p > \frac{N}{2}$. This is exactly the case if the boundary of Ω is sufficiently regular.

THEOREM 3.4 (De Giorgi). *Let Ω be of class C^1, and let f be in $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) with datum f belongs to $C^0(\overline{\Omega})$, and there exists a constant C_p such that*

$$\|u\|_{C^0(\overline{\Omega})} \le C_p\, \|f\|_{L^p(\Omega)}\,.$$

Thanks to the previous result, we thus have the following existence result.

THEOREM 3.5. *Let μ be a measure in $\mathcal{M}(\Omega)$. Then there exists a unique duality solution of (1.8) with datum μ, i.e., a function u in $L^1(\Omega)$ such that*

$$\int_\Omega u\, g = \int_\Omega v\, d\mu, \quad \forall g \in L^\infty(\Omega),$$

where v is the solution of (1.9) with datum g and matrix A^*. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.

4. Regularity of duality solutions

If the datum f belongs to $L^p(\Omega)$, with $1 < p < 2_*$, then the duality solution of (1.9) is more regular.

THEOREM 3.6. *Let f belong to $L^p(\Omega)$, $1 < p < 2_*$. Then the duality solution u of (1.8) belongs to $L^{p^{**}}(\Omega)$, $p^{**} = \frac{Np}{N-2p}$.*

Proof. Let $q = \frac{Np}{Np-N+2p}$, and define $T : L^q(\Omega) \to \mathbb{R}$ as in the proof of Theorem 3.3. We then have

$$|\langle T, g\rangle| \leq \int_\Omega |f|\,|v| \leq \|f\|_{L^p(\Omega)}\ \|v\|_{L^{p'}(\Omega)}\,.$$

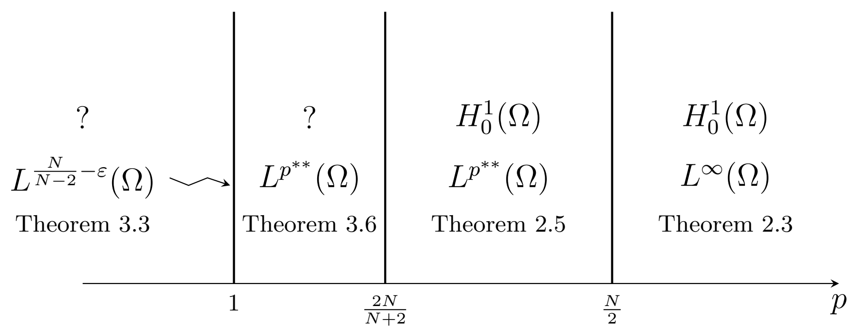
By Theorem 2.5, the norm of v in $L^r(\Omega)$ is controlled by a constant times the norm of g in $L^s(\Omega)$, with $r = s^{**}$. Taking $r = p'$, this gives $s = q$; hence,

$$|\langle T, g\rangle| \leq C\ \|f\|_{L^p(\Omega)}\ \|g\|_{L^q(\Omega)}\,,$$

so that the function u which represents T belongs to $L^{q'}(\Omega)$; since we have $q' = \frac{Np}{N-2p}$, the result is proved. □

Once again, the fact that u belongs to $L^{p^{**}}(\Omega)$ is consistent with the results of Example 2.1 (the case $\frac{N+2}{2} < \alpha < N$).

The picture at the end of Chapter 2 can now be improved as follows.



CHAPTER 4

Existence via approximation for measure data

The result of Theorem 3.5 is somewhat unsatisfactory: even though it proves that there exists a unique solution by duality of (1.9) if the datum belongs to $\mathcal{M}(\Omega)$, it only states that the solution belongs to some Lebesgue space, and does not say anything about the gradient of such a solution. In order to prove gradient estimates on the duality solution we have to proceed in a different way.

THEOREM 4.1. *Let μ belong to $\mathcal{M}(\Omega)$. Then the unique duality solution of (1.8) with datum f belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.*

Proof. Let f_n be a sequence of $L^\infty(\Omega)$ functions which converges to μ in $\mathcal{M}(\Omega)$, with the property that $\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)}$, and let u_n be the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$ and choose $v = T_k(u_n)$ as test function of the weak formulation for u_n. We obtain, recalling that $\nabla u_n = \nabla T_k(u_n)$ where $\nabla T_k(u_n) \neq 0$, and using (1.8),

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n) = \int_\Omega f_n T_k(u_n) \leq k\, \|\mu\|_{\mathcal{M}(\Omega)}\,,$$

where in the last passage we have used that $|T_k(u_n)| \leq k$. Using Sobolev embedding in the left hand side, we have

$$\frac{\alpha}{\mathcal{S}_2^2}\left(\int_\Omega |T_k(u_n)|^{2^*}\right)^{\frac{2}{2^*}} \leq k\, \|\mu\|_{\mathcal{M}(\Omega)}\,.$$

Observing that $|T_k(u_n)| = k$ on the set $A_{n,k} = \{x \in \Omega : |u_n(x)| \geq k\}$, we have

$$\frac{\alpha}{\mathcal{S}_2^2}\, k^2\, (m(A_{n,k}))^{\frac{2}{2^*}} \leq k\, \|\mu\|_{\mathcal{M}(\Omega)}\,,$$

which implies

$$m(A_{n,k}) \leq C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k}\Big)^{\frac{N}{N-2}},$$

with C depending only on N and α. Now we fix $\lambda > 0$, and we have

$$\{|\nabla u_n| \geq \lambda\} = \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup \{|\nabla u_n| \geq \lambda, |u_n| \geq k\},$$

so that

$$\{|\nabla u_n| \geq \lambda\} \subset \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup A_{n,k}.$$

Since

$$m(\{|\nabla u_n| \geq \lambda, |u_n| < k\}) \leq \frac{1}{\lambda^2} \int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{k\ \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2},$$

we have

$$m(\{|\nabla u_n| \geq \lambda\}) \leq \frac{k\ \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2} + C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k}\Big)^{\frac{N}{N-2}},$$

for every $k > 0$. If we choose $k = \lambda^{\frac{N-2}{N-1}}\ \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{N-1}}$, the above inequality becomes

$$m(\{|\nabla u_n| \geq \lambda\}) \leq C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{\lambda}\Big)^{\frac{N}{N-1}}.$$

Let $q < \frac{N}{N-1}$ be fixed, and let $t > 0$. Then

$$\begin{aligned}\int_\Omega |\nabla u_n|^q &= \int_{\{|\nabla u_n|<t\}} |\nabla u_n|^q + \int_{\{|\nabla u_n| \geq t\}} |\nabla u_n|^q \\ &\leq t^q\, m(\Omega) + (q-1) \int_t^{+\infty} \lambda^{q-1}\, m(\{|\nabla u_n| \geq \lambda\})\, d\lambda \\ &\leq t^q\, m(\Omega) + C(q-1)\, \|f\|_{L^1(\Omega)}^{\frac{N}{N-1}} \int_t^{+\infty} \lambda^{q-1-\frac{N}{N-1}}\, d\lambda \\ &= t^q\, m(\Omega) + \frac{C(q-1)}{\frac{N}{N-1} - q} \frac{\|\mu\|_{\mathcal{M}(\Omega)}^{\frac{N}{N-1}}}{t^{\frac{N}{N-1}-q}}.\end{aligned}$$

Choosing $t = \|\mu\|_{\mathcal{M}(\Omega)}$, we obtain

$$\int_\Omega |\nabla u_n|^q \leq C_q\ \|\mu\|_{\mathcal{M}(\Omega)}^q\,, \tag{4.16}$$

so that u_n is bounded in $W_0^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$. Note that C_q diverges as q tends to $\frac{N}{N-1}$. Therefore, up to subsequences, u_n converges to some function u_q weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^1(\Omega)$. Since u_n, being a weak solution, is such that

$$\int_\Omega u_n\, g = \int_\Omega f_n\, v, \quad \forall g \in L^\infty(\Omega),\ \forall n \in \mathbb{N},$$

we can pass to the limit as n tends to infinity to have

$$\int_\Omega u_q\, g = \int_\Omega v\, d\mu, \quad \forall g \in L^\infty(\Omega),$$

so that u_q (which belongs to $W_0^{1,q}(\Omega)$ for *some* $q < \frac{N}{N-1}$) is **the** duality solution of (1.9) with datum μ. This fact is true for *every* $q < \frac{N}{N-1}$, so that u_q does not depend on q. It then follows that the duality solution u of (1.9) belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. □

REMARK 4.2. If $\mu = f$ is a function in $L^1(\Omega)$, and f_n converges to f strongly in $L^1(\Omega)$, we have that f_n is a Cauchy sequence in $L^1(\Omega)$. Thus, if we repeat the proof of the previous theorem working with $u_n - u_m$, using the linearity of the operator, and "keeping track" of $f_n - f_m$, we find that (4.16) becomes

$$\int_\Omega |\nabla u_n - u_m|^q \le C_q \, \|f_n - f_m\|^q_{L^1(\Omega)} \,,$$

for every $q < \frac{N}{N-1}$. Since $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$, it then follows that u_n is a Cauchy sequence in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. This implies that u_n *strongly* converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, so that (up to subsequences) ∇u_n converges to ∇u almost everywhere in Ω.

REMARK 4.3. If $\mu = f$ is a function in $L^1(\Omega)$, and we repeat the proof of the previous theorem working with $u_n - v_n$, where v_n is the solution of (1.9) with a datum g_n which converges to f in $L^1(\Omega)$, we find as before that

$$\int_\Omega |\nabla(u_n - v_n)|^q \le C \, \|f_n - g_n\|^q_{L^1(\Omega)} \,, \tag{4.17}$$

for every $q < \frac{N}{N-1}$. Since $\{f_n - g_n\}$ tends to zero in $L^1(\Omega)$, it then follows that $u_n - v_n$ tends to zero in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. In other words, the solution u found by approximation does not depend on the sequence we choose to approximate the datum f. We already knew this fact (since every approximating sequence converges to the duality solution which is unique), but this different proof may be useful if, for example, the differential operator is not linear, but allows to prove (4.17) in some way, so that the concept of duality solution is not available.

If the datum f is "more regular", one expects solutions with an increased regularity. We already know, from Theorem 3.6, that the summability of u increases with the summability of f, but what happens to the gradient? Recall that if the datum f is "regular" (i.e., if it belongs to $L^{2_*}(\Omega)$), the summability of u increases with that of f, but the gradient of u always belongs to $(L^2(\Omega))^N$. Surprisingly, this is not the case for "bad" solutions, as the following theorem shows.

THEOREM 4.4. *Let f be a function in $L^m(\Omega)$, $1 < m < 2_*$. Then the duality solution of (1.9) belongs to $W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$.*

Proof. Let $f_n = T_n(f)$, and let u_n be the unique solution of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we already know that u_n will converge to the duality solution of (1.9), it is clear that in order to prove the result it will be enough to prove an *a priori* estimate on u_n in $W_0^{1,m^*}(\Omega)$. In order to do that, we fix $h > 0$ and choose $\varphi_h(u_n) = T_1(G_h(u_n))$ as test function in the weak formulation for u_n. If we define $B_h = \{x \in \Omega : h \le |u_n| \le h+1\}$, and $A_h = \{x \in \Omega : |u_n| \ge h\}$ (for the sake of simplicity, we omit the dependence on n on the sets), we obtain, recalling (1.8),

$$\alpha \int_{B_h} |\nabla u_n|^2 \le \int_\Omega A(x)\nabla u_n \cdot \nabla\varphi_h(u_n) = \int_\Omega f_n\,\varphi_h(u_n) \le \int_{A_k} |f|.$$

Let now $0 < \lambda < 1$; we can then write

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} &= \sum_{h=0}^{+\infty} \int_{B_h} \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} \le \sum_{h=0}^{+\infty} \frac{1}{(1+h)^\lambda} \int_{B_h} |\nabla u_n|^2 \\
&\le \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \int_{A_h} |f| = \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \sum_{k=h}^{+\infty} \int_{B_k} |f| \\
&= \sum_{k=0}^{+\infty} \int_{B_k} |f| \sum_{h=0}^{k} \frac{1}{\alpha(1+h)^\lambda} \\
&\le C \sum_{k=0}^{+\infty} \int_{B_k} |f|\,(1+k)^{1-\lambda} \le C \int_\Omega |f|(1+|u_n|)^{1-\lambda} \\
&\le C\,\|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{1}{m'}}.
\end{aligned}$$

Let now $q > 1$ be fixed. Then, by Sobolev and Hölder inequality,

$$\begin{aligned}
\frac{1}{\mathcal{S}_q^q} \left(\int_\Omega |u_n|^{q^*} \right)^{\frac{q}{q^*}} &\le \int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\lambda\frac{q}{2}}}\,(1+|u_n|)^{\lambda\frac{q}{2}} \\
&\le \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} \right)^{\frac{q}{2}} \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}} \\
&\le C\,\|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{q}{2m'}} \\
&\quad \times \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}}.
\end{aligned}$$

We now choose λ and q in such a way that

$$(1-\lambda)m' = q^* = \frac{\lambda q}{2-q}.$$

This implies

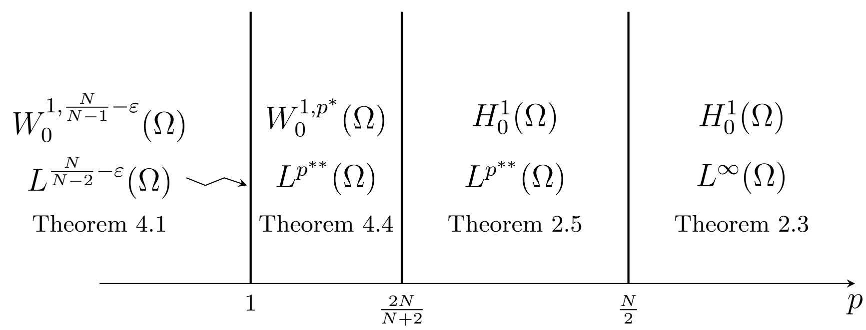
$$\lambda = \frac{N(2-q)}{N-q}, \quad q = m^* = \frac{Nm}{N-m}.$$

It is easy to see that $1 < m < 2_*$ implies $0 < \lambda < 1$, as desired. We thus have

$$\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{q}{q^*}} \leq C \int_\Omega |\nabla u_n|^q \leq C \, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{q^*}\right)^{1-\frac{q}{2m}}.$$

Since $\frac{q}{q^*} > 1 - \frac{q}{2m}$ is true (being equivalent to $m < \frac{N}{2}$), we obtain from the first and third term that u_n is bounded in $L^{q^*}(\Omega)$ (which is again $L^{m^{**}}(\Omega)$, see Theorem 2.5) by a constant depending (among other quantities) on the norm of f in $L^m(\Omega)$. Once u_n is bounded, the boundedness of $|\nabla u_n|$ in $L^q(\Omega)$ (with $q = m^*$) then follows comparing the second and the third term. ☐

We can now draw the complete picture.



CHAPTER 5

Nonuniqueness for distributional solutions

If the datum μ is a measure, we have proved in Theorem 4.1 that the sequence u_n of approximating solutions is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Therefore, and up to subsequences, u_n weakly converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Choosing a $C_0^1(\Omega)$ test function φ in the formulation (1.10) for u_n, we obtain

$$\int_\Omega A(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega f_n\,\varphi,$$

which, passing to the limit, yields

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi = \int_\Omega \varphi\,d\mu \quad \forall\varphi \in C_0^1(\Omega),$$

so that u is a solution *in the sense of distributions* of (1.9). Since the definition of solution in the sense of distributions can always be given (even when the notion of duality solution is unavailable due for example to the operator being nonlinear), one may wonder whether there is a way of proving uniqueness of distributional solutions (not passing through duality solutions).

The following example is due to J. Serrin (see [**7**]). Let $\varepsilon > 0$ and $A^\varepsilon(x)$ be the symmetric matrix defined by

$$a_{ij}^\varepsilon(x) = \delta_{ij} + (a_\varepsilon - 1)\frac{x_i\,x_j}{|x|^2}.$$

If $a_\varepsilon = \frac{N-1}{\varepsilon(N-2+\varepsilon)}$, then the function

$$w^\varepsilon(x) = x_1\,|x|^{1-N-\varepsilon}$$

is a solution in the sense of distributions of

$$-\mathrm{div}(A^\varepsilon(x)\,\nabla w^\varepsilon) = 0, \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.18}$$

Indeed, if we rewrite $w(x) = x_1|x|^\alpha$ and

$$a_{ij}(x) = \delta_{ij} + \beta\frac{x_i\,x_j}{|x|^2},$$

simple (but tedious) calculations imply

$$w_{x_1}(x) = |x|^\alpha + \alpha x_1^2|x|^{\alpha-2}, \quad w_{x_i}(x) = \alpha x_1 x_i|x|^{\alpha-2},$$

so that

$$\sum_{i=1}^{N} a_{ij}(x)\, w_{x_i}(x) = \delta_{1j}|x|^{\alpha} + (\alpha\beta+\alpha+\beta)x_1 x_j |x|^{\alpha-2}.$$

Therefore,

$$(A(x)\,\nabla w)_{x_1} = \alpha x_1|x|^{\alpha-2} + (\alpha\beta+\alpha+\beta)[2x_1|x|^{\alpha-2} + (\alpha-2)x_1^3|x|^{\alpha-4}],$$

and

$$(A(x)\,\nabla w)_{x_j} = (\alpha\beta+\alpha+\beta)[x_1|x|^{\alpha-2} + (\alpha-2)x_1 x_j^2|x|^{\alpha-4}],$$

so that

$$\mathrm{div}(A(x)\nabla w) = x_1|x|^{\alpha-2}[\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta)].$$

Given $0<\varepsilon<1$, if we choose $\alpha = 1-N-\varepsilon$, and $\beta = \frac{N-1}{\varepsilon(N-2+\varepsilon)}+1$, we have

$$\alpha + (N-1+\alpha)(\alpha\beta+\alpha+\beta) = 0,$$

so that w is a solution of (5.18) if $x \neq 0$. To prove that w^ε is a solution in the sense of distributions in the whole $\mathbb{R}^N$, let φ be a function in $C_0^1(\Omega)$, and observe that since $|A^\varepsilon(x)\nabla w^\varepsilon|$ belongs to $L^1(\Omega)$, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon \cdot \nabla\varphi = \lim_{r\to 0^+}\int_{\mathbb{R}^N\setminus B_r(0)} A^\varepsilon(x)\nabla w^\varepsilon\cdot\nabla\varphi.$$

Using Gauss-Green formula, and recalling that w^ε is a solution of the equation outside the origin, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon\cdot\nabla\varphi = -\lim_{r\to 0^+}\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma,$$

where ν is the exterior normal to $B_r(0)$, i.e., $\nu = \frac{x}{r}$. By a direct computation,

$$A^\varepsilon(x)\nabla w^\varepsilon\cdot\frac{x}{r} = Qx_1|r|^{\alpha-1},$$

with $Q = 1+\alpha\beta+\alpha+\beta = -\frac{N-1}{\varepsilon}$. Therefore, recalling the value of α, and rescaling to the unit sphere,

$$-\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma = \frac{N-1}{\varepsilon}\frac{1}{r^\varepsilon}\int_{\partial B_1(0)}\varphi(ry)x_1\, d\sigma.$$

Using again the Gauss-Green formula, we have

$$\int_{\partial B_1(0)}\varphi(ry)x_1\, d\sigma = r\int_{B_1(0)} e_1\cdot\nabla\varphi(rx)\, dx,$$

where $e_1 = (1,0,\dots,0)$. Therefore, since $0<\varepsilon<1$, we have

$$\lim_{r\to 0^+}\int_{\partial B_r(0)}\varphi\, A^\varepsilon(x)\nabla w^\varepsilon\cdot\nu\, d\sigma = \lim_{r\to 0^+} r^{1-\varepsilon}\int_{B_1(0)} e_1\cdot\nabla\varphi(rx)\, dx = 0,$$

so that w^ε is a solution in the sense of distributions of $-\text{div}(A^\varepsilon \nabla w^\varepsilon) = 0$ in the whole $\mathbb{R}^N$.

Let now $\Omega = B_1(0)$ be the unit ball, and let v_ε be the unique solution of

$$\begin{cases} -\text{div}(A^\varepsilon(x)\,\nabla v^\varepsilon) = \text{div}(A^\varepsilon(x)\,\nabla x_1) & \text{in } \Omega, \\ v^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists since $\text{div}(A^\varepsilon(x)\,\nabla x_1)$ is a regular function belonging to $H^{-1}(\Omega)$ (as can be easily seen). Therefore, the function $z^\varepsilon = v^\varepsilon + x_1$ is the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} -\text{div}(A^\varepsilon(x)\,\nabla z^\varepsilon) = 0 & \text{in } \Omega, \\ z^\varepsilon = x_1 & \text{on } \partial\Omega, \end{cases}$$

so that the function $u^\varepsilon = w^\varepsilon - z^\varepsilon$ is a solution in the sense of distributions of

$$\begin{cases} -\text{div}(A^\varepsilon(x)\,\nabla u^\varepsilon) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which is not identically zero since z^ε belongs to $H^1(\Omega)$, while w^ε belongs to $W_0^{1,q}(\Omega)$ for every $q < q_\varepsilon = \frac{N}{N-1+\varepsilon}$. Hence, the problem

$$\begin{cases} -\text{div}(A^\varepsilon(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has infinitely many solutions in the sense of distributions, which can be written as $u = \overline{u} + t\,u^\varepsilon$, t in $\mathbb{R}$, where $\overline{u}$ is the duality solution.

One may observe that the solution found by approximation belongs to $W_0^{1,q}(\Omega)$ for *every* $q < \frac{N}{N-1}$, while the solution of the above example belongs to $W_0^{1,q}(\Omega)$ for *some* $q < \frac{N}{N-1}$, and that we are not allowed to take $\varepsilon = 0$ since in this case a_ε diverges. Thus one may hope that there is still uniqueness of the solution obtained by approximation. However, it is possible to modify Serrin's example in dimension $N \geq 3$ (see [**6**]) to find a nonzero solution in the sense of distributions for

$$\begin{cases} -\text{div}(B^\varepsilon(x)\,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Here

$$B^\varepsilon(x) = \begin{pmatrix} 1 + (a_\varepsilon - 1)\frac{x_1^2}{x_1^2+x_2^2} & (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 0 \\ (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 1 + (a_\varepsilon - 1)\frac{x_2^2}{x_1^2+x_2^2} & 0 \\ 0 & 0 & I \end{pmatrix},$$

where I is the identity matrix in $\mathbb{R}^{N-2}$, and a_ε is as above, with ε fixed so that $w^\varepsilon(x) = x_1\,(\sqrt{x_1^2 + x_2^2})^{\varepsilon-1}$ belongs to $W^{1,q}(\mathbb{R}^2)$ for every $q < 2$.

On the other hand, in dimension $N = 2$ there is a unique solution in the sense of distributions belonging to $W_0^{1,q}(\Omega)$, for every $q < 2$. The proof of this fact uses Meyers' regularity theorem for linear equations with regular data.

THEOREM 5.1 (Meyers). *Let A be a matrix which satisfies (1.8). Then there exists $p > 2$ (p depends on the ratio $\frac{\alpha}{\beta}$ and becomes larger as $\frac{\alpha}{\beta}$ tends to 1) such that if u is a solution of (1.9) with datum f belonging to $L^\infty(\Omega)$, then u belongs to $W_0^{1,p}(\Omega)$.*

THEOREM 5.2. *Let $N = 2$. Then there exists a unique solution in the sense of distributions of (1.9) such that u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$.*

Proof. Since the equation is linear, it is enough to prove that if u is such that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla\varphi = 0, \quad \forall\varphi \in C_0^1(\Omega),$$

then $u = 0$. Since u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$, it is enough to prove that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla\varphi = 0, \quad \forall\varphi \in W_0^{1,p}(\Omega),$$

for some $p > 2$, implies $u = 0$. Let B be a subset of Ω, and let v_B be the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v_B) = \chi_B & \text{in } \Omega, \\ \qquad v = 0 & \text{on } \partial\Omega. \end{cases}$$

By Meyers' theorem, v_B belongs to $W_0^{1,p}(\Omega)$, for some $p > 2$. Hence

$$\int_\Omega A(x)\,\nabla u \cdot \nabla v_B = 0,$$

while, choosing u as test function in the weak formulation for v_B (which can be done using a density argument and the regularity of ∇v_B), we have

$$\int_\Omega A^*(x)\,\nabla v_B \cdot \nabla u = \int_B u.$$

Therefore,

$$\int_B u = 0, \quad \forall B \subseteq \Omega,$$

and this implies $u \equiv 0$. □

CHAPTER 6

Entropy solutions

As we have seen, uniqueness of solutions for distributional solutions can fail even in the linear case if the regularity of the solutions is not "enough" to allow the choice of less regular test functions. And the lack of regularity of the solution of the counterexample by Serrin (as modified in [**6**]) is exactly the one which is typical of the solutions of equations with data in $L^1(\Omega)$ or in $\mathcal{M}(\Omega)$. In the linear case, however, the lack of uniqueness is avoided by using the concept of duality solution, but it is enough for the operator to be non linear (say, $-\operatorname{div}(a(x,u)\nabla u)$, with a a bounded function) in order to "lose" the duality definition. This problem is much more evident for operators which are nonlinear also with respect to the gradient. In this case, a further condition on the solutions has been looked for, in order to guarantee uniqueness (at least for the solutions obtained by approximation).

The first remark about solutions obtained by approximation is the following (see the proof of Theorem 4.1): even though the solutions do not belong to $H_0^1(\Omega)$ (since they belong to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$), the *truncates* of the solutions are in the "energy space" $H_0^1(\Omega)$, and satisfy the following estimate

$$\alpha \int_\Omega |\nabla T_k(u)|^2 \leq k \, \|\mu\|_{\mathcal{M}(\Omega)} \,.$$

In other words, the solutions are not in $H_0^1(\Omega)$ "only" where they become "infinite". Since the function in the counterexample of Serrin has not the truncates in the energy space $H_0^1(\Omega)$, one may think that the "correct" space where to look for uniqueness of solutions is the following:

$$\mathcal{T}_0^{1,2}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable: } T_k(u) \in H_0^1(\Omega), \ \forall k > 0\}.$$

This set of functions has a further property: that every function in it has, in some sense, a "gradient".

THEOREM 6.1. *Let u belong to $\mathcal{T}_0^{1,2}(\Omega)$. Then there exists a unique (up to a.e. equivalence) measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$v\,\chi_{\{|u| \leq k\}} = \nabla T_k(u), \quad \text{a.e. in } \Omega, \, \forall k > 0.$$

Proof. In order to prove the result, it is enough to prove that the function v, defined as $\nabla T_k(u)$ on the set $\{|u| \leq k\}$, does not depend on k. Let $\overline{x}$ in Ω be such that $u(\overline{x}) = k$. Then $\overline{x}$ belongs to the set $\{|u(x)| \leq k + \varepsilon\}$ for every $\varepsilon \geq 0$. Therefore, by definition,

$$v(\overline{x}) = \nabla T_{k+\varepsilon}(u(\overline{x})), \quad \forall \varepsilon \geq 0.$$

On the other hand, $T_{k+\varepsilon}(T_k(u)) = T_k(u)$, so that

$$\nabla T_{k+\varepsilon}(T_k(u)) = \nabla T_k(u),$$

which implies

$$\nabla T_{k+\varepsilon}(u(\overline{x})) = \nabla T_{k+\varepsilon}(T_k(u(\overline{x}))) = \nabla T_k(u(\overline{x})),$$

so that the value of v does not depend on ε. □

From now, we will *define* the gradient ∇u of a function u in $\mathcal{T}_0^{1,2}(\Omega)$ as the function v given by the previous theorem. It is easy to see that if u belongs to $W_0^{1,1}(\Omega)$, then the function v given by the theorem is nothing but the "standard" distributional gradient of u.

Remark that $\mathcal{T}_0^{1,2}(\Omega)$ is not a vector space: there exist functions u and v in $\mathcal{T}_0^{1,2}(\Omega)$ such that $u+v$ does not belong to the same space. If however u, v and $u+v$ are in $\mathcal{T}_0^{1,2}(\Omega)$, then we also have $\nabla(u+v) = \nabla u + \nabla v$.

Even though the space $\mathcal{T}_0^{1,2}(\Omega)$ seems the natural one where to look for solutions, this is not the case: the fact that u in $\mathcal{T}_0^{1,2}(\Omega)$ is a solution in the sense of distributions is not enough in order to prove that it is unique. In order to do that we need something more (and also the fact that the datum belongs to $L^1(\Omega)$), following [**1**].

Definition 6.2. Let f be in $L^1(\Omega)$. A function u in $\mathcal{T}_0^{1,2}(\Omega)$ is an *entropy solution* of (1.9) if

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - \varphi) \leq \int_\Omega f\, T_k(u - \varphi), \tag{6.19}$$

for every $k > 0$ and for every φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 6.3. Every term in (6.19) is well defined. The right hand side is finite since $T_k(u-\varphi)$ belongs to $L^\infty(\Omega)$, while the left hand side is well defined since $\nabla T_k(u-\varphi)$ is different from zero only if $|u-\varphi| \leq k$. On this set, $|u| \leq k + \|\varphi\|_{L^\infty(\Omega)} = M$, so that we have

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - \varphi) = \int_{\{|u-\varphi|\leq k\}} A(x)\nabla T_M(u) \cdot \nabla(T_M(u) - \varphi),$$

which is finite since u belongs to $\mathcal{T}_0^{1,2}(\Omega)$ and φ belongs to $H_0^1(\Omega)$.

We now prove an existence result for entropy solutions.

THEOREM 6.4 (see [**1**]). *Let f be an $L^1(\Omega)$ function. Then there exists an entropy solution u of (1.9).*

Proof. As usual, we work by approximation. Let $f_n = T_n(f)$, and let u_n be the solution of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$. Taking $T_k(u_n)$ as test function we obtain, using (1.8),

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n) = \int_\Omega f_n T_k(u_n) \leq k \, \|f\|_{L^1(\Omega)} ,$$

so that the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$ for fixed k. This implies that there exists a function v_k in $H_0^1(\Omega)$ such that, up to subsequences, $T_k(u_n)$ converges to v_k weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. From Remark 4.2 we know that u_n converges to u (the unique duality solution of (1.9)) strongly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, and that ∇u_n converges to ∇u almost everywhere in Ω. This implies that $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^2(\Omega)$, and so $v_k = T_k(u)$. Thus, by Fatou lemma,

$$\alpha \int_\Omega |\nabla T_k(u)|^2 \leq \liminf_{n\to+\infty} \alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq k \, \|f\|_{L^1(\Omega)} ,$$

which implies that u belongs to $\mathcal{T}_0^{1,2}(\Omega)$. We now fix $k > 0$, φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and choose $v = T_k(u_n - \varphi)$ as test function in the weak formulation (1.10) of (1.9), and we have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n - \varphi) = \int_\Omega f_n \, T_k(u_n - \varphi).$$

For the right hand side we have, by Lebesgue theorem,

$$\lim_{n\to+\infty} \int_\Omega f_n \, T_k(u_n - \varphi) = \int_\Omega f \, T_k(u - \varphi),$$

while the left hand side can be rewritten as

$$\int_\Omega A(x)\nabla T_k(u_n - \varphi) \cdot \nabla T_k(u_n - \varphi) + \int_\Omega A(x)\nabla \varphi \cdot \nabla T_k(u_n - \varphi).$$

The first term is nonnegative, so that the almost everywhere convergence of ∇u_n to ∇u implies, by Fatou lemma,

$$\int_\Omega A(x)\nabla T_k(u-\varphi)\cdot\nabla T_k(u-\varphi) \leq \liminf_{n\to+\infty} \int_\Omega A(x)\nabla T_k(u_n-\varphi)\cdot\nabla T_k(u_n-\varphi),$$

while for the second we have (thanks to the weak convergence of $T_k(u_n)$ to $T_k(u)$ in $H_0^1(\Omega)$)

$$\int_\Omega A(x)\nabla\varphi\cdot\nabla T_k(u-\varphi) = \lim_{n\to+\infty}\int_\Omega A(x)\nabla\varphi\cdot\nabla T_k(u_n-\varphi).$$

Putting together these results, and cancelling equal terms, we have

$$\int_\Omega A(x)\nabla u\cdot\nabla T_k(u-\varphi) \leq \int_\Omega f\,T_k(u-\varphi),$$

so that u is an entropy solution of (1.9). □

THEOREM 6.5. *Let f be an $L^1(\Omega)$ function, and let u be an entropy solution of (1.9) with datum f. Then u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, and is a distributional solution of (1.9).*

Proof. Taking $\varphi = 0$ in (6.19) we obtain, recalling (1.8),

$$\alpha\int_\Omega |\nabla T_k(u)|^2 \leq \int_\Omega A(x)\nabla u\cdot\nabla T_k(u) = \int_\Omega f\,T_k(u) \leq k\,\|f\|_{L^1(\Omega)}\,.$$

From this inequality we can reason as in the proof of Theorem 4.1 to obtain (4.16) for u, so that u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$.

We now fix $h > 0$ and choose $\varphi = T_h(u)$ as test function in (6.19). We obtain

$$\int_\Omega A(x)\nabla u\cdot\nabla T_k(u-T_h(u)) \leq \int_\Omega f\,T_k(u-T_h(u)),$$

which can be rewritten as

$$\int_{\{h-k\leq|u|\leq h+k\}} A(x)\nabla u\cdot\nabla u = \int_{\{|u|\geq h\}} f\,T_k(u-T_h(u)) \leq k\int_{\{|u|\geq h\}}|f|.$$

Defining $A_h = \{|u| \geq h\}$, we have that $m(A_h)$ tends to zero as h tends to infinity (since u belongs to $W_0^{1,1}(\Omega)$, hence to $L^1(\Omega)$). Since f belongs to $L^1(\Omega)$, we have

$$\lim_{h\to+\infty}\int_{\{|u|\geq h\}}|f| = 0,$$

so that, recalling (1.8)

$$\lim_{h\to+\infty}\int_{\{h-k\leq|u|\leq h+k\}}|\nabla u|^2 = 0. \tag{6.20}$$

Let now $h > 0$ and ψ in $C_0^1(\Omega)$ be fixed, and choose $\varphi = T_h(u)-\psi$ as test function in the entropy formulation (6.19) written for $k = \|\psi\|_{L^\infty(\Omega)}$. We obtain

$$\int_\Omega A(x)\nabla u\cdot\nabla T_k(u-T_h(u)+\psi) \leq \int_\Omega f\,T_k(u-T_h(u)+\psi).$$

Using Lebesgue theorem, and the choice of k, it is easy to see that

$$\lim_{h\to+\infty}\int_\Omega f\,T_k(u-T_h(u)+\psi)=\int_\Omega f\,T_k(\psi)=\int_\Omega f\,\psi.$$

As for the left hand side, using again the choice of k, we can rewrite it as

$$\int_{\{|u|\le h\}}A(x)\nabla u\cdot\nabla\psi+\int_{\{|u|\ge h\}}A(x)\nabla u\cdot\nabla T_k(u-T_h(u)+\psi).$$

Since A is bounded, u belongs to $W_0^{1,1}(\Omega)$ (actually, even better), and ψ is in $C_0^1(\Omega)$, we have (by Lebesgue theorem)

$$\lim_{h\to+\infty}\int_{\{|u|\le h\}}A(x)\nabla u\cdot\nabla\psi=\int_\Omega A(x)\nabla u\cdot\nabla\psi.$$

On the other hand, since (again by the choice of k)

$$\{|u-T_h(u)+\psi|\le k,\ |u|\ge h\}\subseteq\{h-2k\le|u|\le h+2k\},$$

we have, by (1.8),

$$\left|\int_{\{|u|\ge h\}}A(x)\nabla u\cdot\nabla T_k(u-T_h(u)+\psi)\right|$$
$$\le\beta\int_{\{h-2k\le|u|\le h+2k\}}|\nabla u|(|\nabla u|+|\nabla\psi|),$$

so that by (6.20) we have

$$\lim_{h\to+\infty}\int_{\{|u|\ge h\}}A(x)\nabla u\cdot\nabla T_k(u-T_h(u)+\psi)=0.$$

Putting together the results, we obtain

$$\int_\Omega A(x)\nabla u\cdot\nabla\psi\le\int_\Omega f\,\psi,$$

for every ψ in $C_0^1(\Omega)$. Exchanging ψ with $-\psi$ we obtain the reverse inequality so that u is a distributional solution of (1.9). □

Not only an entropy solution exists, it is also unique.

THEOREM 6.6. *Let f be a function in $L^1(\Omega)$. Then the entropy solution of (1.9) is unique.*

Proof. We present three proofs of this result.
1) An entropy solution is a duality solution. We fix g in $L^\infty(\Omega)$, and let v be the solution of

$$\begin{cases}-\mathrm{div}(A^*(x)\,\nabla v)=g & \text{in }\Omega,\\ \qquad v=0 & \text{on }\partial\Omega.\end{cases}$$

By Theorem 2.3, v belongs to $L^\infty(\Omega)$. Now we repeat the proof of Theorem 6.5, choosing $\varphi = T_h(u) - v$ in the entropy formulation, with $h > 0$ and $k = \|v\|_{L^\infty(\Omega)}$. We obtain

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - T_h(u) + v) \le \int_\Omega f\, T_k(u - T_h(u) + v).$$

As before, we have (by Lebesgue theorem and by the choice of k)

$$\lim_{h\to+\infty} \int_\Omega f\, T_k(u - T_h(u) + v) = \int_\Omega f\, v,$$

and as before the left hand side can be rewritten as

$$\int_{\{|u|\le h\}} A(x)\nabla u \cdot \nabla v + \int_{\{|u|\ge h\}} A(x)\nabla u \cdot \nabla T_k(u - T_h(u) + v).$$

For the second term we can reason as in the proof of Theorem 6.5 to have (using (6.20)) that

$$\lim_{h\to+\infty} \int_{\{|u|\ge h\}} A(x)\nabla u \cdot \nabla T_k(u - T_h(u) + v) = 0,$$

while the first can be rewritten as

$$\begin{aligned}\int_{\{|u|\le h\}} A(x)\nabla u \cdot \nabla v &= \int_\Omega A(x)\nabla T_h(u) \cdot \nabla v \\ &= \int_\Omega A^*(x)\nabla v \cdot \nabla T_h(u) = \int_\Omega g\, T_h(u),\end{aligned}$$

since $T_h(u)$, being in $H_0^1(\Omega)$, can be chosen as test function in the problem solved by v. Thus, by Lebesgue theorem,

$$\lim_{h\to+\infty} \int_{\{|u|\le h\}} A(x)\nabla u \cdot \nabla v = \int_\Omega g\, u.$$

Putting together the results, we have

$$\int_\Omega g\, u \le \int_\Omega f\, v.$$

Exchanging g with $-g$ (and so v with $-v$, by linearity), we obtain the reverse inequality, so that u is a duality solution of (1.9).

2) An entropy solution is a solution obtained by approximation. Here we follow [**5**]. Let f_n be a sequence of $L^\infty(\Omega)$ functions that converges to f in $L^1(\Omega)$, and let u_n be the solution of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 2.3, u_n belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, so that $\varphi = u_n$ is an admissbile choice in the entropy formulation for u. We then have

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - u_n) \le \int_\Omega f\, T_k(u - u_n).$$

On the other hand, $T_k(u - u_n)$ belongs to $H_0^1(\Omega)$, and so it can be chosen as test function in the weak formulation for u_n. We then have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u - u_n) = \int_\Omega f_n\, T_k(u - u_n).$$

By subtracting the above results, we have

$$\int_\Omega A(x)\nabla(u - u_n) \cdot \nabla T_k(u - u_n) \le \int_\Omega (f - f_n)\, T_k(u - u_n),$$

and using (1.8) we obtain

$$\alpha \int_\Omega |\nabla T_k(u - u_n)|^2 \le k\, \|f - f_n\|_{L^1(\Omega)}\,.$$

Letting n tend to infinity, we have that $T_k(u - u_n)$ tends to zero in $H_0^1(\Omega)$, and this implies that u_n converges to the entropy solution u. Since solutions obtained by approximation are unique, the entropy solution u is unique.
3) There exists at most an entropy solution. Here we follow [**1**]. Let u and v be two entropy solutions of (1.9), with the same datum f, and let $h > k > 0$. Then $\varphi = T_h(v)$ is admissible in the entropy formulation for u, and $\varphi = T_h(u)$ is admissible in the entropy formulation for u. We thus obtain

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - T_h(v)) \le \int_\Omega f\, T_k(u - T_h(v)),$$

and

$$\int_\Omega A(x)\nabla v \cdot \nabla T_k(v - T_h(u)) \le \int_\Omega f\, T_k(v - T_h(u)).$$

Summing the two inequalities, we obtain

$$\int_\Omega A(x)\nabla u \cdot \nabla T_k(u - T_h(v)) + \int_\Omega A(x)\nabla v \cdot \nabla T_k(v - T_h(u))$$

in the left hand side, and

$$\int_\Omega f(T_k(u - T_h(v)) + T_k(v - T_h(u)))$$

in the right hand side. Since $T_k(s)$ is an odd function, we obtain, by Lebesgue theorem,

$$\lim_{h\to+\infty} \int_\Omega f(T_k(u - T_h(v)) + T_k(v - T_h(u))) = 0,$$

so that

$$\limsup_{h\to+\infty} \int_\Omega A(x)\nabla u\cdot\nabla T_k(u-T_h(v)) + \int_\Omega A(x)\nabla v\cdot\nabla T_k(v-T_h(u)) \leq 0.$$

For the sake of simplicity we will suppose from now on that $u \geq 0$ and $v \geq 0$, since the proof turns out to be considerably simplified. We refer to [**1**] for the proof in the general case of changing sign solutions. We write

$$\Omega = \{u \leq h,\ v \leq h\} \cup \{u > h,\ v \leq h\} \cup \{v > h\} = E_0^h \cup F_1^h \cup F_2^h,$$

and

$$\Omega = \{v \leq h,\ u \leq h\} \cup \{v > h,\ u \leq h\} \cup \{u > h\} = E_0^h \cup F_3^h \cup F_4^h.$$

We then have

$$\int_{E_0^h} A(x)\nabla u\cdot\nabla T_k(u-T_h(v)) = \int_{E_0^h} A(x)\nabla u\cdot\nabla T_k(u-v),$$

and, analogously,

$$\int_{E_0^h} A(x)\nabla v\cdot\nabla T_k(v-T_h(u)) = \int_{E_0^h} A(x)\nabla v\cdot\nabla T_k(v-u),$$

On the other hand,

$$\int_{F_1^h} A(x)\nabla u\cdot\nabla T_k(u-T_h(v)) = \int_{\left\{\substack{u>h,\ v\leq h\\ 0\leq u-v\leq k}\right\}} A(x)\nabla u\cdot\nabla(u-v);$$

on the set $\{u > h,\ v \leq h,\ 0 \leq u-v \leq k\}$ we have both $h < u \leq h+k$ and $h-k < v \leq h$, so that

$$\left|\int_{F_1^h} A(x)\nabla u\cdot\nabla T_k(u-T_h(v))\right| \leq \beta \int_{\left\{\substack{h<u\leq h+k\\ h-k<v\leq h}\right\}} |\nabla u||\nabla v|.$$

Using (6.20) for both u and v we have

$$\lim_{h\to+\infty} \int_{\{h<u\leq h+k\}} |\nabla u|^2 = 0 = \lim_{h\to+\infty} \int_{\{h-k<v\leq h\}} |\nabla v|^2,$$

so that, by Hölder inequality, we have

$$\lim_{h\to+\infty} \left|\int_{F_1^h} A(x)\nabla u\cdot\nabla T_k(u-T_h(v))\right| = 0;$$

repeating the same proof, we have

$$\lim_{h\to+\infty} \left|\int_{F_3^h} A(x)\nabla v\cdot\nabla T_k(v-T_h(u))\right| = 0.$$

Furthermore,

$$\int_{F_2^h} A(x)\nabla u \cdot \nabla T_k(u - T_h(v)) = \int_{\left\{ \substack{v>h \\ 0 \le u < h+k} \right\}} A(x)\nabla u \cdot \nabla u \ge 0,$$

and analogously

$$\int_{F_4^h} A(x)\nabla v \cdot \nabla T_k(v - T_h(u)) = \int_{\left\{ \substack{u>h \\ 0 \le v < h+k} \right\}} A(x)\nabla v \cdot \nabla v \ge 0.$$

Putting the results together, we have

$$\limsup_{h\to+\infty} \int_{E_0^h} A(x)\nabla(u-v) \cdot \nabla T_k(u-v) \le 0,$$

which, by Fatou lemma, implies, since E_0^h "fills" Ω as h tends to infinity,

$$0 \le \int_\Omega A(x)\nabla(u-v) \cdot \nabla T_k(u-v) \le 0.$$

This, and (1.8), imply $\nabla T_k(u-v) \equiv 0$, and so $u = v$. □

What happens if the datum f is the Dirac mass concentrated at one point in Ω? In this case the definition of entropy solution is no longer enough to guarantee its uniqueness, as the following example shows.

EXAMPLE 6.7. Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^N$, $N \ge 3$, and let $u(x)$ be the unique duality solution of

$$\begin{cases} -\Delta u = \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.21}$$

It is well known that $u(x) = u(|x|) = \frac{|x|^{2-N}-1}{(N-2)\omega_N}$, and that u is the limit of the sequence u_n of solutions of

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f_n = \frac{N n^N}{\omega_N}$ in the ball $B_{\frac{1}{n}}(0)$, and $f_n = 0$ elsewhere. Since u_n is radially symmetric, it can be easily calculated, obtaining that

$$u_n(x) = \begin{cases} u(x) & \text{in } B_1(0) \setminus B_{\frac{1}{n}}(0), \\ -\frac{n^N}{2\omega_n}|x|^2 + \frac{N n^{N-2}-2}{2(N-2)\omega_N} & \text{in } B_{\frac{1}{n}}(0). \end{cases}$$

If φ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, k is fixed and n is large enough, we thus have $T_k(u_n - \varphi) = k$ in $B_{\frac{1}{n}}(0)$, and so

$$\int_\Omega \nabla u_n \cdot \nabla T_k(u_n - \varphi) = \int_\Omega f_n \, T_k(u_n - \varphi) = k.$$

Passing to the limit on n as in the proof of Theorem 6.4, we thus have

$$\int_\Omega \nabla u \cdot \nabla T_k(u-\varphi) \le k = T_k(u-\varphi)(0) = \int_\Omega T_k(u-\varphi)\, d\delta_0,$$

so that u is an entropy solution of (6.21). Observe that

$$\int_\Omega T_k(u-\varphi)\, d\delta_0$$

is well defined since $T_k(u-\varphi)$ is continuous (being constantly equal to k) in a neighbourhood of the origin. Let now $\lambda < 1$, and consider the function $u_\lambda = \lambda u$. We have

$$\int_\Omega \nabla u_\lambda \cdot \nabla T_k(u_\lambda - \varphi) = \lambda^2 \int_\Omega \nabla u \cdot \nabla T_{\frac{k}{\lambda}}(u - \frac{\varphi}{\lambda}) \le \lambda^2 \frac{k}{\lambda} = \lambda\, k \le k,$$

so that also u_λ is an entropy solution of (6.21).

If, instead of passing to the limit as in the proof of Theorem 6.4 (i.e., dropping a nonnegative term), one performs explicit calculations, one finds that u is such that

$$\int_\Omega \nabla u \cdot \nabla T_k(u-\varphi) = k = \int_\Omega T_k(u-\varphi)\, d\delta_0, \tag{6.22}$$

for every φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$; in other words, the duality solution of (6.21) is an entropy solution *with equality sign*, while of course u_λ is an entropy solution *with inequality sign* for every $0 < \lambda < 1$. Therefore, one may wonder whether uniqueness can be recovered for measure data by requiring that u is an entropy solution "with equality sign". Indeed, this is not the case; to see it, let v_a be the duality solution of

$$\begin{cases} -\Delta v_a = \delta_a & \text{in } \Omega, \\ \quad v_a = 0 & \text{on } \partial\Omega, \end{cases}$$

where a is a point in Ω (different from the origin). Performing the same calculations as above, we find that

$$\int_\Omega \nabla v_a \cdot \nabla T_k(v_a - \varphi) = k = \int_\Omega T_k(v_a - \varphi) d\delta_a,$$

for every $k > 0$ and for every φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Consider now $w = u + v_a$. Clearly w is the duality solution of

$$\begin{cases} -\Delta w = \delta_0 + \delta_a & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases}$$

so that, once again, for every $k > 0$ and for every φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \nabla w \cdot \nabla T_k(w - \varphi) = 2k = \int_\Omega T_k(w - \varphi) d(\delta_0 + \delta_a).$$

However, since w "explodes" both at the origin and at a, we have (if $0 < \lambda < 2$)

$$\int_\Omega T_k(w - \varphi) d(\lambda \delta_0 + (2 - \lambda) d\delta_a) = \lambda k + (2 - \lambda) k = 2k,$$

so that $w = u + v_a$ is an entropy solution, with equality sign, of the equation

$$\begin{cases} -\Delta w = \lambda \delta_0 + (2 - \lambda)\delta_a & \text{in } \Omega, \\ \qquad w = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation, however, also has as entropy solution with equality sign its duality solution $z = \lambda u + (2 - \lambda) v_a$ (or any other linear combination of u and v_a with coefficients ν and $2 - \nu$, $0 < \nu < 2$).

REMARK 6.8. Let f belong to $L^1(\Omega)$, and let u be the (duality, entropy, found by approximation) solution of (1.9). If we use the entropy formulation, written for $\varphi = 0$, we have

$$\int_\Omega A(x) \nabla T_k(u) \cdot \nabla T_k(u) = \int_\Omega A(x) \nabla u \cdot \nabla T_k(u) \le \int_\Omega f\, T_k(u).$$

Dividing by k, and then letting k tend to infinity, we have, by (1.8) and by Lebesgue theorem,

$$0 \le \lim_{k \to +\infty} \frac{1}{k} \int_\Omega A(x) \nabla T_k(u) \cdot \nabla T_k(u) \le \lim_{k \to +\infty} \int_\Omega f\, \frac{T_k(u)}{k} = 0.$$

In other words,

$$\lim_{k \to +\infty} \frac{1}{k} \int_{\{|u| \le k\}} A(x) \nabla u \cdot \nabla u = 0. \tag{6.23}$$

If, instead of $\varphi = 0$, we choose $\varphi = T_k(u)$, we find (with the same calculations)

$$\lim_{k \to +\infty} \frac{1}{k} \int_{\{k \le |u| \le 2k\}} A(x) \nabla u \cdot \nabla u = 0. \tag{6.24}$$

This formula, and (6.23), state that, even though the quantities

$$\int_{\{|u| \le k\}} A(x) \nabla u \cdot \nabla u \quad \text{and} \quad \int_{\{k \le |u| \le 2k\}\}} A(x) \nabla u \cdot \nabla u$$

do not remain bounded as k tends to infinity (since the solution u does not belong to $H_0^1(\Omega)$, but to a larger space), a suitable "rescaling" of them not only remains bounded, but converges to zero.

If, instead of taking an $L^1(\Omega)$ function, we consider a Dirac mass as datum, this fact is no longer true. Take for example $\Omega = B_1(0)$ and consider the unique duality solution of

$$\begin{cases} -\Delta u = \delta_0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

As stated in Example 6.7, we have $u(x) = u(|x|) = \frac{|x|^{2-N}-1}{(N-2)\omega_N}$. If we fix $k > 0$, then

$$T_k(u) = \begin{cases} u & \text{in } B_1(0) \setminus B_{r_k}(0), \\ k & \text{in } B_{r_k}(0), \end{cases}$$

where r_k is such that $\frac{r_k^{2-N}-1}{(N-2)\omega_N} = k$. If we calculate the "energy" of $\nabla T_k(u)$, we have

$$\int_{\{|u|\leq k\}} |\nabla u|^2 = \int_{B_1(0)\setminus B_{r_k}(0)} |\nabla u|^2 = \frac{\omega_N}{\omega_N^2} \int_{r_k}^1 \rho^{1-N}\, d\rho = \frac{r_k^{2-N}-1}{(N-2)\omega_N} = k,$$

so that, even though the "rescaled" energy is bounded, we have

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{|u|\leq k\}} |\nabla u|^2 = 1.$$

An analogous calculation yields

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{k\leq|u|\leq 2k\}} |\nabla u|^2 = 1.$$

CHAPTER 7

Decomposition of measures using capacity

What is the difference between a measure in $\mathcal{M}(\Omega)$ and a function in $L^1(\Omega)$? For example, between a Dirac mass concentrated at the origin and the function $\frac{1}{|x|^N \log^2(|x|)}$? As we have seen, both the Dirac mass and f yield a solution which only belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, but in the case of the $L^1(\Omega)$ datum a certain "energy", when renormalized, tends to zero (while it is constant for the Dirac mass). While the vanishing of the renormalized energy happens for any $L^1(\Omega)$ datum, one may wonder for which measures the "Dirac mass" phenomenon happens. Before answering to this question we need some (more!) tools.

1. Capacity

Given a subset E of Ω, we define the (harmonic) *capacity* of E as

$$\operatorname{cap}(E) = \inf\left\{\int_\Omega |\nabla\varphi|^2,\ \varphi \in H_0^1(\Omega),\ \varphi \geq \chi_E\right\}.$$

The set function $\operatorname{cap}(\cdot)$ is *not* a measure on Ω, nor it is bounded (if E "touches" the boundary of Ω the set of functions in $H_0^1(\Omega)$ greater than χ_E is empty, so that the infimum is $+\infty$). It is however a monotone and σ-subadditive set function, in the sense that

$$\operatorname{cap}\Big(\bigcup_{n=1}^{+\infty} E_n\Big) \leq \sum_{n=1}^{+\infty} \operatorname{cap}(E_n).$$

If E is an open subset of Ω, then the infimum in the definition of $\operatorname{cap}(E)$ is actually a minimum, which is achieved on an $H_0^1(\Omega)$ function u_E, which satisfies $0 \leq u_E \leq 1$ in Ω. If K is compact in Ω, then $\operatorname{cap}(K)$ can be obtained by taking the infimum of the "energy" of φ over the functions φ in $C_0^1(\Omega)$ which are larger than χ_K.

Recalling the Sobolev embedding, if φ is a function in $H_0^1(\Omega)$ which is larger than χ_E, we have

$$\mathcal{L}^N(E) \leq \int_E |\varphi|^{2^*} \leq \int_\Omega |\varphi|^{2^*} \leq \mathcal{S}_2^2 \left(\int_\Omega |\nabla\varphi|^2\right)^{\frac{2}{2^*}}.$$

Taking the infimum on the right hand side, we thus obtain

$$\mathcal{L}^N(E) \le \mathcal{S}_2^2\,(\mathrm{cap}(E))^{\frac{N-2}{N}},$$

so that sets of zero capacity have also zero Lebesgue measure. As a matter of fact, sets of zero capacity are "thinner" than sets of zero Lebesgue measure: they have Hausdorff dimension smaller than $N-2$.

Even though capacity is not a measure, one can always decompose measures with respect to it.

THEOREM 7.1. *Let μ be a measure in $\mathcal{M}(\Omega)$. Then there exists a unique pair (μ_0, λ) such that $\mu = \mu_0 + \lambda$, and*

$$\mathrm{cap}(B) = 0 \Rightarrow \mu_0(B) = 0, \quad \lambda = \mu \llcorner E, \text{ with } \mathrm{cap}(E) = 0.$$

Proof. See the proof of Theorem 3.1, and remark that we only used the σ-subadditivity of ν in order to prove it. □

Since every set of zero capacity has zero Lebesgue measure, it is clear that if f belongs to $L^1(\Omega)$ then the measure μ defined by

$$\mu(B) = \int_B f(x)\,dx$$

is such that $\mathrm{cap}(B) = 0$ implies $\mu(E) = 0$, so that $\mu_0 = \mu$. On the other hand, if $\mu = \delta_{x_0}$, the Dirac mass concentrated at x_0, then since $\mathrm{cap}(\{x_0\}) = 0$, we have that μ is singular with respect to capacity, and so $\lambda = \mu$. There is however another set of measures such that $\mu_0 = \mu$.

THEOREM 7.2. *Let μ be a nonnegative measure in $H^{-1}(\Omega)$, i.e., a measure such that there exists T in $H^{-1}(\Omega)$ for which*

$$\langle T, \varphi\rangle = \int_\Omega \varphi\,d\mu, \quad \forall \varphi \in H_0^1(\Omega).$$

Then $\mathrm{cap}(B) = 0$ implies $\mu(B) = 0$.

Proof. Since $\mathrm{cap}(B) = 0$, there exists a sequence φ_ε in $H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla\varphi_\varepsilon|^2 \le \varepsilon, \quad \varphi_\varepsilon \ge \chi_B.$$

Let u be the solution of

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists since μ belongs to $H^{-1}(\Omega)$. Taking φ_ε as test function, we obtain (recalling that $\mu \ge 0$ and using Hölder inequality),

$$0 \le \mu(B) \le \int_\Omega \varphi_\varepsilon\,d\mu = \int_\Omega \nabla u \cdot \nabla\varphi_\varepsilon \le C\sqrt{\varepsilon},$$

so that, letting ε tend to zero, $\mu(B) = 0$. □

As a consequence of the previous theorem, if we define the set of "soft measures"

$$\mathcal{M}_0(\Omega) = \{\mu \in \mathcal{M}(\Omega) : \mu(B) = 0 \ \forall B : \mathrm{cap}(B) = 0\},$$

and the set of "singular measures"

$$\mathcal{M}_{\mathrm{s}}(\Omega) = \{\mu \in \mathcal{M}(\Omega) : \mu = \mu \llcorner E \text{ with } \mathrm{cap}(E) = 0\},$$

we have that $L^1(\Omega)+H^{-1}(\Omega) \subseteq \mathcal{M}_0(\Omega)$. Note that there exist functions in $L^1(\Omega)$ which are not in $H^{-1}(\Omega)$ (for example, $f(x) = \frac{1}{|x|^N \log^2(|x|)}$), and measures in $H^{-1}(\Omega)$ which are not in $L^1(\Omega)$ (for example, the $(N-1)$-dimensional Hausdorff measure restricted on an hypersurface of codimension 1 in $\mathbb{R}^N$), but that the intersection of $L^1(\Omega)$ with $H^{-1}(\Omega)$ is not $\{0\}$ (for example, $L^{2_*}(\Omega)$ is a subset of both spaces). The "nice" fact is that the opposite inclusion holds.

THEOREM 7.3. *Let μ be a measure in $\mathcal{M}(\Omega)$. Then μ belongs to $L^1(\Omega) + H^{-1}(\Omega)$ if and only if μ belongs to $\mathcal{M}_0(\Omega)$.*

Proof. See [**2**]. □

Therefore, given a measure μ in $\mathcal{M}(\Omega)$, we can first decompose it (uniquely) as

$$\mu = \mu_0 + \lambda, \quad \mu_0 \in \mathcal{M}_0(\Omega),\ \lambda \in \mathcal{M}_{\mathrm{s}}(\Omega),$$

and then we can further decompose it (not uniquely, as far as μ_0 is concerned) as

$$\mu = f + T + \lambda^+ - \lambda^-, \quad f \in L^1(\Omega),\ T \in H^{-1}(\Omega),\ \lambda^\pm \in \mathcal{M}_{\mathrm{s}}(\Omega).$$

The question is now the following: we have uniqueness of entropy solutions for $L^1(\Omega)$ data, and we have uniqueness of solutions (hence of entropy solutions) for $H^{-1}(\Omega)$ data (by Lax-Milgram). Thus, by linearity, we have uniqueness of entropy solutions for data in $\mathcal{M}_0(\Omega)$. We know that if the datum is δ_0 we have counterexamples to uniqueness due to the nonvanishing of a certain renormalized energy (see Remark 6.8), and we know that δ_0 belongs to $\mathcal{M}_{\mathrm{s}}(\Omega)$. Is the renormalized energy nonvanishing for every measure in $\mathcal{M}_{\mathrm{s}}(\Omega)$, or is the Dirac mass a special case?

CHAPTER 8

Renormalized solutions

The result obtained in Remark 6.8 can be improved: not only a certain renormalized energy remains constant as k diverges if the datum is a Dirac mass, but we can also "recover" the datum from it.

As stated in Remark 6.8, if u is the duality solution of $-\Delta u = \delta_0$, i.e.,

$$u(x) = \frac{|x|^{2-N} - 1}{(N-2)\omega_N},$$

then the sequence $\frac{|\nabla u|^2}{k}\chi_{\{|u|\leq k\}}$ is bounded in $L^1(\Omega)$ (since it has "mass" equal to 1 for every k), so that (up to subsquences) it converges to some measure λ in the weak* topology of measures. We are going to prove that $\lambda = \delta_0$, so that the measure datum can be in some sense "reconstructed" by a suitable rescaling of the "energy" of the solution. If φ is a fixed continuous function, we have

$$\frac{1}{k}\int_{\{|u|\leq k\}} |\nabla u|^2\,\varphi = \frac{1}{k\omega_N^2}\int_{B_1(0)\setminus B_{r_k}(0)} |x|^{2-2N}\,\varphi(x)\,dx,$$

where r_k is such that $u(r_k) = k$. Passing to spherical coordinates we have

$$\frac{1}{k}\int_{\{|u|\leq k\}} |\nabla u|^2\,\varphi = \frac{1}{k\omega_N^2}\int_{S^{N-1}}\int_{r_k}^{1} \rho^{1-N}\varphi(\rho,\sigma)\,d\rho\,d\sigma.$$

Defining $y = \frac{\rho}{r_k}$, we then have

$$\frac{1}{k}\int_{\{|u|\leq k\}} |\nabla u|^2\,\varphi = \frac{r_k^{2-N}}{k\omega_N^2}\int_{S^{N-1}}\int_{1}^{\frac{1}{r_k}} y^{1-N}\varphi(r_k y,\sigma)\,dy\,d\sigma.$$

Since $r_k y$ tends to zero for every fixed y in $(1,+\infty)$, since φ is continuous, and since y^{1-N} belongs to $L^1(1,+\infty)$, we have (by Lebesgue theorem)

$$\lim_{k\to+\infty}\int_{1}^{\frac{1}{r_k}} y^{1-N}\varphi(r_k y,\sigma)\,dy = \varphi(0)\int_{1}^{+\infty} y^{1-N}\,dy = \frac{\varphi(0)}{N-2};$$

on the other hand, by definition of r_k,

$$\lim_{k\to+\infty} \frac{r_k^{2-N}}{k\omega_N^2} = \frac{N-2}{\omega_N},$$

so that

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{|u|\le k\}} |\nabla u|^2 \, \varphi = \frac{\varphi(0)}{\omega_N} \int_{S^{N-1}} d\sigma = \varphi(0),$$

as desired. A similar calculation yields

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{k\le|u|\le 2k\}} |\nabla u|^2 \, \varphi = \varphi(0).$$

We will use this fact to give a new definition of solution for (1.9).

1. Renormalized solutions

The results of this section are contained in [**4**].

Definition 8.1. A function u in $L^1(\Omega)$ is a *renormalized solution* of

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = \mu = \mu_0 + \lambda^+ - \lambda^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, and if

$$\begin{aligned} \int_\Omega A(x)\nabla u \cdot \nabla(h(u)\varphi) = & \int_\Omega h(u)\varphi \, d\mu_0 \\ & + h^{+\infty} \int \varphi \, d\lambda^+ - h^{-\infty} \int \varphi \, d\lambda^-, \end{aligned} \tag{8.25}$$

for every h in $W^{1,\infty}(\mathbb{R})$ such that $\mathrm{supp}(h')$ is compact, and for every φ in $C_0^1(\Omega)$; here $h^{\pm\infty}$ are the limits of h at $\pm\infty$, respectively. If $h(0) = 0$, one may choose φ in $C^1(\Omega)$, while if h has compact support, one may choose φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

What is the meaning of (8.25)? It more or less says that $h(u) = h^{+\infty}$, λ^+ almost everyhwere, and that $h(u) = h^{-\infty}$, λ^- almost everyhwere, which means that $u = \pm\infty$ on the support of λ. If we take $h(s) = T_k(s)$, and suppose that $\mu = f + \lambda$, with f in $L^1(\Omega)$ and λ singular and nonnegative, we obtain

$$\int_\Omega A(x)\nabla u \cdot \nabla(T_k(u)\varphi) = \int_\Omega f \, T_k(u)\varphi + k \int_\Omega \varphi \, d\lambda,$$

which can be rewritten as

$$\frac{1}{k}\int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)\,\varphi+\frac{1}{k}\int_\Omega A(x)\nabla u\cdot\nabla\varphi\,T_k(u)$$
$$=\frac{1}{k}\int_\Omega f\,T_k(u)\varphi+\int_\Omega\varphi\,d\lambda.$$

Letting k tend to infinity, the second and third term converge to zero, so that

$$\lim_{k\to+\infty}\frac{1}{k}\int_{\{|u|\le k\}}A(x)\nabla u\cdot\nabla u\,\varphi=\int_\Omega\varphi\,d\lambda,$$

which means that a renormalized solution allows to “reconstruct” the singular part of the datum as limit of some rescaled energy (as happened for the laplacian and for the Dirac mass as datum).

We are going to prove an existence and uniqueness result for renormalized solutions of (1.9) with nonnegative data. In order to simplify the presentation, we will limit ourselves to the case of a nonnegative datum μ of the form $\mu=f+\lambda$, with f a nonnegative function in $L^1(\Omega)$, and λ a nonnegative bounded Radon measure concentrated on a set E of zero harmonic capacity. The (much longer, and involved) proof can be found in [**4**].

To prove the result, we need to define suitable cut-off functions.

THEOREM 8.2. *Let λ be a nonnegative measure in $\mathcal{M}_{\rm s}(\Omega)$, concentrated on a set E of zero capacity. Then, for every $\delta>0$ there exist a function ψ_δ in $C^1_0(\Omega)$ such that $0\le\psi_\delta\le 1$,*

$$\int_\Omega|\nabla\psi_\delta|^2\le\delta,$$

and

$$0\le\int_\Omega(1-\psi_\delta)d\lambda\le\delta. \tag{8.26}$$

Proof. See [**4**]. □

We will consider a sequence u_n of solutions in $H^1_0(\Omega)\cap L^\infty(\Omega)$ of

$$\begin{cases}-\mathrm{div}(A(x)\nabla u_n)=f_n+\lambda_n & \text{in }\Omega,\\ u_n=0 & \text{on }\partial\Omega,\end{cases} \tag{8.27}$$

where f_n is a sequence of nonnegative functions in $L^\infty(\Omega)$ that converges to f in $L^1(\Omega)$, and λ_n is a sequence of nonnegative functions in $L^\infty(\Omega)$ that converges to λ in the narrow topology of measures, i.e. such that

$$\lim_{n\to+\infty}\int_\Omega f_n\,\varphi=\int_\Omega\varphi\,d\lambda,\quad\forall\varphi\in C^0_{\rm b}(\Omega).$$

The solution u_n of (8.27) exists and is unique, and is nonnegative since the datum $f_n + \lambda_n$ is nonnegative.

In the next result we recall some of the properties of the sequence u_n.

THEOREM 8.3. *The sequence u_n is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, and the sequence $T_k(u_n)$ is bounded in $H_0^1(\Omega)$. Furthermore, there exists a function u belonging to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, such that (up to subsequences) u_n converges to u weakly in $W_0^{1,q}(\Omega)$; moreover, u_n and ∇u_n converge to u and ∇u almost everywhere in Ω, respectively.*

Proof. The fact that $T_k(u_n)$ is bounded in $H_0^1(\Omega)$, and that u_n is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, follows from the result of Theorem 4.1. From standard compactness results for Sobolev spaces it then follows that (up to subsequences) u_n weakly converges to some u in $W_0^{1,q}(\Omega)$, and that it converges to the same function almost everywhere in Ω. Therefore, it only remains to prove the almost everywhere convergence of ∇u_n to ∇u. Observe that Remark 4.2 is no longer applicable since λ_n is not a Cauchy sequence in $L^1(\Omega)$.

In order to prove this result, we are going to prove that

$$\lim_{n\to+\infty} \int_\Omega [A(x)\nabla(u_n - u)\cdot\nabla(u_n - u)]^{\frac{1}{2}} = 0, \tag{8.28}$$

which, in view of (1.8), will imply the convergence of ∇u_n to ∇u in $(L^1(\Omega))^N$ (hence the almost everywhere convergence of ∇u_n to ∇u up to subsequences). If we define

$$\Psi_n(x) = A(x)\nabla(u_n - u)\cdot\nabla(u_n - u),$$

then, if $k > 0$ is given,

$$\int_\Omega \Psi_n(x) = \int_{\{|u|\le k\}} \Psi_n(x) + \int_{\{|u|>k\}} \Psi_n(x) = I_{n,k} + J_{n,k}.$$

We have, recalling (1.8), and choosing $1 < q < \frac{N}{N-1}$,

$$J_{n,k} \le \beta^{\frac{1}{2}} 2^{1-\frac{1}{q}} \left(\int_{\{|u|>k\}} (|\nabla u_n|^q + |\nabla u|^q) \right)^{\frac{1}{q}} m(\{|u| > k\})^{1-\frac{1}{q}}.$$

Since u_n is bounded in $W_0^{1,q}(\Omega)$, and u belongs to the same space, we thus have

$$0 \le J_{n,k} \le C\, m(\{|u| > k\})^{1-\frac{1}{q}}.$$

Since u belongs to $L^1(\Omega)$, this implies

$$\lim_{k\to+\infty} \lim_{n\to+\infty} J_{n,k} = 0. \tag{8.29}$$

For $I_{n,k}$ we have, since $\Psi_n(x)$ is nonnegative,

$$
\begin{aligned}
I_{n,k} &= \int_{\{|u|\le k\}} [A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u))]^{\frac{1}{2}} \\
&\le \int_\Omega [A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u))]^{\frac{1}{2}} = L_{n,k}.
\end{aligned}
$$

Given $\varepsilon > 0$ we then have

$$
\begin{aligned}
L_{n,k} &= \int_{\{|u_n - T_k(u)|\le \varepsilon\}} [A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u))]^{\frac{1}{2}} \\
&\quad + \int_{\{|u_n - T_k(u)|> \varepsilon\}} [A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u))]^{\frac{1}{2}} \\
&= M_{n,k,\varepsilon} + N_{n,k,\varepsilon}.
\end{aligned}
$$

For $N_{n,k,\varepsilon}$ we have, using again the boundedness of u_n in $W_0^{1,q}(\Omega)$, and the fact that u belongs to the same space,

$$
N_{n,k,\varepsilon} \le C\, m(\{|u_n - T_k(u)| > \varepsilon\})^{1-\frac{1}{q}},
$$

which implies, since u_n converges almost everywhere to u,

$$
\lim_{\varepsilon\to 0^+} \lim_{k\to+\infty} \lim_{n\to+\infty} N_{n,k,\varepsilon} = 0. \tag{8.30}
$$

For $M_{n,k,\varepsilon}$ we have

$$
M_{n,k,\varepsilon} \le \left(\int_{\{|u_n - T_k(u)|\le\varepsilon\}} A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u)) \right)^{\frac{1}{2}} m(\Omega)^{\frac{1}{2}},
$$

so that we only have to deal with

$$
P_{n,k,\varepsilon} = \int_{\{|u_n - T_k(u)|\le\varepsilon\}} A(x)\nabla(u_n - T_k(u)) \cdot \nabla(u_n - T_k(u)),
$$

which we rewrite as

$$
\begin{aligned}
P_{n,k,\varepsilon} &= \int_\Omega A(x)\nabla(u_n - T_k(u)) \cdot \nabla T_\varepsilon(u_n - T_k(u)) \\
&= \int_\Omega A(x)\nabla u_n \cdot \nabla T_\varepsilon(u_n - T_k(u)) \\
&\quad - \int_\Omega A(x)\nabla T_k(u) \cdot \nabla T_\varepsilon(u_n - T_k(u)) = Q_{n,k,\varepsilon} + R_{n,k,\varepsilon}.
\end{aligned}
$$

We have, thanks to the fact that $T_k(u_n)$ weakly converges to $T_k(u)$ in $H_0^1(\Omega)$,

$$
\lim_{n\to+\infty} R_{n,k,\varepsilon} = \int_\Omega A(x)\nabla T_k(u) \cdot \nabla T_\varepsilon(u - T_k(u)) = 0,
$$

since $u - T_k(u) = 0$ on the set $\{|u| \leq k\}$ where $\nabla T_k(u)$ is different from zero. Thus,

$$\lim_{\varepsilon\to 0^+} \lim_{k\to+\infty} \lim_{n\to+\infty} R_{n,k,\varepsilon} = 0. \tag{8.31}$$

To deal with $Q_{n,k,\varepsilon}$, we use (at last!) the equation, to obtain

$$Q_{n,k,\varepsilon} = \int_\Omega (f_n + \lambda_n)\, T_\varepsilon(u_n - T_k(u)) \leq \varepsilon\,(\|f_n\|_{L^1(\Omega)} + \|\lambda_n\|_{L^1(\Omega)}),$$

which implies

$$\lim_{\varepsilon\to 0^+} \lim_{k\to+\infty} \lim_{n\to+\infty} Q_{n,k,\varepsilon} = 0. \tag{8.32}$$

Putting together (8.29), (8.30), (8.31) and (8.32) we thus have (8.28), as desired. □

REMARK 8.4. One may wonder why there is need to prove the almost everywhere convergence of the gradients of u_n: the equation being linear, boundedness in a Sobolev space is enough to guarantee weak convergence, hence passage to the limit in the approximate equations. However, if we take $h(u_n)\,\varphi$ as test function in (8.27), with h and φ as in the definition of renormalized solution, we have

$$\int_\Omega A(x)\nabla u_n\cdot\nabla u_n h'(u_n)\varphi + \int_\Omega A(x)\nabla u_n\cdot\nabla\varphi h(u_n) = \int_\Omega (f_n+\lambda_n)h(u_n)\varphi,$$

and it is clear that to pass to the limit as n tends to infinity in the first term we need the strong convergence of u_n in some space (to be precise, since h' has compact support, the strong convergence of the truncates of u_n); using weak convergence, we will only obtain an inequality (as is for entropy solutions), which is not enough for our purposes. And indeed, we are going to prove that $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$.

LEMMA 8.5. *Let g_n be a sequence of nonnegative functions in $L^1(\Omega)$ that converges almost everywhere to some function g in $L^1(\Omega)$. If*

$$\lim_{n\to+\infty} \int_\Omega g_n = \int_\Omega g, \tag{8.33}$$

then g_n converges to g in $L^1(\Omega)$.

Proof. We have

$$\int_\Omega |g_n - g| = \int_\Omega (g_n - g) + 2\int_\Omega (g - g_n)\chi_{\{0\leq g_n\leq g\}}.$$

The first term tends to zero by (8.33), while the second one tends to zero by Lebesgue theorem, since $(g - g_n)\chi_{\{0 \le g_n \le g\}}$ tends to zero almost everywhere, and

$$|(g - g_n)\chi_{\{0 \le g_n \le g\}}| \le 2g,$$

with $2g$ in $L^1(\Omega)$. □

THEOREM 8.6. *Let u_n be the solution of (8.27), and suppose that it satisfies all the properties stated in Theorem 8.3. Then $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$.*

In what follows, we will denote by $\omega(n, j, \delta)$ a quantity such that

$$\lim_{\delta \to 0^+} \lim_{j \to +\infty} \lim_{n \to +\infty} \omega(n, j, \delta) = 0.$$

Should we not take one or more of the limits, we will only write the dependance of $\omega(\cdot)$ from the variables that go to the limit. Analogously, we will denote by $\omega_j(n)$ and $\omega_{j,\delta}(n)$ some quantities such that

$$\lim_{n \to +\infty} \omega_j(n) = 0, \qquad \lim_{n \to +\infty} \omega_{j,\delta}(n) = 0,$$

for every $j > 0$ and for every $\delta > 0$.

Proof. We will split the proof in several steps.
Step 1. Let $\delta > 0$, and let ψ_δ be given by Theorem 8.2. Then

$$0 \le \frac{1}{j} \int_{\{j \le u_n \le 2j\}} A(x)\nabla u_n \cdot \nabla u_n (1 - \psi_\delta) \le \omega(n, j, \delta). \tag{8.34}$$

Let $\beta_j(s) = \frac{1}{j} T_j(G_j(s))$, and choose $\beta_j(u_n)(1 - \psi_\delta)$ as test function in (8.27); we have

$$\begin{aligned} \frac{1}{j} \int_{\{j \le u_n \le 2j\}} & A(x)\nabla u_n \cdot \nabla u_n (1 - \psi_\delta) - \int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta \beta_j(u_n) \\ &= \int_\Omega f_n \beta_j(u_n)(1 - \psi_\delta) + \int_\Omega \lambda_n \beta_j(u_n)(1 - \psi_\delta). \end{aligned}$$

Since ψ_δ is in $C_0^1(\Omega)$, u_n is bounded in $W_0^{1,q}(\Omega)$ for some $q > 1$, and $\beta_j(u_n)$ converges weakly* in $L^\infty(\Omega)$ to $\beta_j(u)$, we have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta \beta_j(u_n) = \int_\Omega A(x)\nabla u \cdot \nabla \psi_\delta \beta_j(u) + \omega_{j,\delta}(n);$$

since $\beta_j(u)$ tends to zero weakly* in $L^\infty(\Omega)$ as j tends to infinity we then have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta \beta_j(u_n) = \omega_\delta(n, j).$$

Analogously (since f_n converges to f in $L^1(\Omega)$), we have

$$\int_\Omega f_n\beta_j(u_n)(1-\psi_\delta) = \omega_\delta(n,j).$$

As far as the last term is concerned, we have, since $0 \le \beta_j(u_n) \le 1$,

$$0 \le \int_\Omega \lambda_n\beta_j(u_n)(1-\psi_\delta) \le \int_\Omega \lambda_n(1-\psi_\delta).$$

Since λ_n converges to λ in the narrow topology of measures, we have, by (8.26)

$$\int_\Omega \lambda_n(1-\psi_\delta) = \int_\Omega (1-\psi_\delta)d\lambda + \omega_\delta(n) \le \omega(n,\delta).$$

Putting together the results found so far, we obtain (8.34).

Step 2. Let $\delta > 0$ and let ψ_δ be the function given by Theorem 8.2. Then, for every $k > 0$,

$$\int_\Omega A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)\psi_\delta = \omega(n,\delta), \tag{8.35}$$

and

$$\int_\Omega (k-T_k(u_n))\psi_\delta\lambda_n = \omega(n,\delta). \tag{8.36}$$

In order to prove this result, we choose $(k-T_k(u_n))\psi_\delta$ as test function in (8.27), to find (recalling that $u_n \ge 0$)

$$\begin{aligned} -\int_\Omega & A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)\psi_\delta \\ &+\int_\Omega A(x)\nabla u_n\cdot\nabla\psi_\delta(k-T_k(u_n)) \\ &= \int_\Omega f_n(k-T_k(u_n))\psi_\delta + \int_\Omega \lambda_n(k-T_k(u_n))\psi_\delta. \end{aligned}$$

For the second term we have, since $k - T_k(u_n)$ is different from zero only where $0 \le u_n \le k$,

$$\int_\Omega A(x)\nabla u_n\cdot\nabla\psi_\delta(k-T_k(u_n)) = \int_\Omega A(x)\nabla T_k(u_n)\cdot\nabla\psi_\delta(k-T_k(u_n)).$$

The weak convergence of $T_k(u_n)$ to $T_k(u)$ in $H_0^1(\Omega)$ (as well as the $L^\infty(\Omega)$ weak* convergence of $k-T_k(u_n)$ to $k-T_k(u)$), allows us to pass to the limit on n, obtaining

$$\int_\Omega A(x)\nabla u_n\cdot\nabla\psi_\delta(k-T_k(u_n)) = \int_\Omega A(x)\nabla T_k(u)\cdot\nabla\psi_\delta(k-T_k(u)) + \omega_\delta(n).$$

Since $T_k(u)$ belongs to $H_0^1(\Omega)$, $k - T_k(u)$ belongs to $L^\infty(\Omega)$, and ψ_δ tends to zero in the same space, we then have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta (k - T_k(u_n)) = \omega(n, \delta).$$

The fact the f_n converges to f in $L^1(\Omega)$ (together with the $L^\infty(\Omega)$ weak* convergence of $k - T_k(u_n)$ to $k - T_k(u)$) implies that

$$\int_\Omega f_n (k - T_k(u_n))\psi_\delta = \int_\Omega f(k - T_k(u))\psi_\delta + \omega_\delta(n),$$

so that the $L^\infty(\Omega)$ weak* convergence of ψ_δ to zero implies

$$\int_\Omega f_n (k - T_k(u_n))\psi_\delta = \omega(n, \delta).$$

Using these results, we therefore have

$$\int_\Omega A(x)\nabla T_k(u_n) \cdot \nabla T_k(u_n)\psi_\delta + \int_\Omega \lambda_n (k - T_k(u_n))\psi_\delta = \omega(n, \delta),$$

so that (8.35) and (8.36) follow observing that both terms above are nonnegative.

Step 3. For every $k > 0$ we have

$$\begin{aligned} &\int_\Omega A(x)\nabla T_k(u_n) \cdot \nabla T_k(u_n)(1 - \psi_\delta) \\ &\quad = \int_\Omega A(x)\nabla T_k(u) \cdot \nabla T_k(u)(1 - \psi_\delta) + \omega(n, \delta). \end{aligned} \tag{8.37}$$

In order to prove (8.37), we begin by proving that

$$\begin{aligned} &\int_\Omega A(x)\nabla T_k(u_n) \cdot \nabla T_k(u_n)(1 - \psi_\delta) - \int_\Omega A(x)\nabla u \cdot \nabla \psi_\delta T_k(u) \\ &\quad = \int_\Omega f\, T_k(u)(1 - \psi_\delta) + \omega(n, \delta). \end{aligned} \tag{8.38}$$

To do this, we choose $T_k(u_n)(1-\psi_\delta)$ as test function in (8.27), to obtain

$$\begin{aligned} &\int_\Omega A(x)\nabla T_k(u_n) \cdot \nabla T_k(u_n)(1 - \psi_\delta) - \int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta T_k(u_n) \\ &\quad = \int_\Omega f_n T_k(u_n)(1 - \psi_\delta) + \int_\Omega \lambda_n T_k(u_n)(1 - \psi_\delta). \end{aligned}$$

The fact that u_n converges to u in $W_0^{1,q}(\Omega)$ (for some $q > 1$), that $T_k(u_n)$ converges to $T_k(u)$ in the weak* topology of $L^\infty(\Omega)$, and that ψ_δ is in $C_0^1(\Omega)$ implies

$$\int_\Omega A(x)\nabla u_n \cdot \nabla \psi_\delta T_k(u_n) = \int_\Omega A(x)\nabla u \cdot \nabla \psi_\delta T_k(u) + \omega_\delta(n).$$

Analogously, we have

$$\int_\Omega f_n T_k(u_n)(1-\psi_\delta) = \int_\Omega f T_k(u)(1-\psi_\delta) + \omega_\delta(n),$$

while, by (8.26),

$$0 \le \int_\Omega \lambda_n T_k(u_n)(1-\psi_\delta) \le k \int_\Omega (1-\psi_\delta) d\lambda + \omega_\delta(n) = \omega(n,\delta).$$

Putting together the results, we find (8.38). The proof of (8.37) will be complete once we prove that

$$\begin{aligned}\int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)(1-\psi_\delta) - \int_\Omega A(x)\nabla u\cdot\nabla\psi_\delta T_k(u) \\ = \int_\Omega f\, T_k(u)(1-\psi_\delta) + \omega(\delta).\end{aligned} \tag{8.39}$$

In order to do that, we choose $(1-\beta_j(u_n))T_k(u)(1-\psi_\delta)$ as test function in (8.27), where $\beta_j(s)$ has been defined in Step 1. We have

$$\begin{aligned}-\frac{1}{j}\int_{\{j\le u_n\le 2j\}} & A(x)\nabla u_n\cdot\nabla u_n T_k(u)(1-\psi_\delta) \\ &+ \int_\Omega A(x)\nabla u_n\cdot\nabla T_k(u)(1-\beta_j(u_n))(1-\psi_\delta) \\ &- \int_\Omega A(x)\nabla u_n\cdot\nabla\psi_\delta(1-\beta_j(u_n))T_k(u) \\ = &\int_\Omega f_n(1-\beta_j(u_n))T_k(u)(1-\psi_\delta) \\ &+ \int_\Omega \lambda_n(1-\beta_j(u_n))T_k(u)(1-\psi_\delta).\end{aligned}$$

For the first term we have, by (8.34),

$$\frac{1}{j}\int_{\{j\le u_n\le 2j\}} A(x)\nabla u_n\cdot\nabla u_n T_k(u)(1-\psi_\delta) \le k\,\omega(n,j,\delta).$$

For the second term, since $1-\beta_j(u_n)$ is different from zero only where $0\le u_n\le 2j$, we have

$$\begin{aligned}\int_\Omega & A(x)\nabla u_n\cdot\nabla T_k(u)(1-\beta_j(u_n))(1-\psi_\delta) \\ &= \int_\Omega A(x)\nabla u\cdot\nabla T_k(u)(1-\beta_j(u))(1-\psi_\delta) + \omega_{j,\delta}(n) \\ &= \int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)(1-\beta_j(u))(1-\psi_\delta) + \omega_{j,\delta}(n).\end{aligned}$$

Since $T_k(u)$ belongs to $H_0^1(\Omega)$, and since $1-\beta_j(u)$ tends to 1 in $L^\infty(\Omega)$ weak*, we then have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u)(1-\beta_j(u_n))(1-\psi_\delta) \\ = \int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)(1-\psi_\delta) + \omega_\delta(n,j).$$

For the third term we have, since u_n converges to u weakly in $W_0^{1,q}(\Omega)$ (for some $q>1$), and since ψ_δ belongs to $C_0^1(\Omega)$,

$$\int_\Omega A(x)\nabla u_n\cdot\nabla\psi_\delta(1-\beta_j(u_n))T_k(u) \\ = \int_\Omega A(x)\nabla u\cdot\nabla\psi_\delta(1-\beta_j(u))T_k(u) + \omega_{j,\delta}(n) \\ = \int_\Omega A(x)\nabla u\cdot\nabla\psi_\delta T_k(u) + \omega_\delta(n,j),$$

where in the last passage we have used again that $\beta_j(u)$ tends to 1 in $L^\infty(\Omega)$ weak*. For the fourth term we have

$$\int_\Omega f_n(1-\beta_j(u_n))T_k(u)(1-\psi_\delta) \\ = \int_\Omega f(1-\beta_j(u))T_k(u)(1-\psi_\delta) + \omega_{j,\delta}(n) \\ = \int_\Omega fT_k(u)(1-\psi_\delta) + \omega_\delta(n,j),$$

while for the fifth term we have, by (8.26),

$$0 \le \int_\Omega \lambda_n(1-\beta_j(u_n))T_k(u)(1-\psi_\delta) \le \int_\Omega \lambda_n(1-\psi_\delta) = \omega(n,\delta).$$

Putting together the results, we obtain (8.39). Comparing (8.38) and (8.39) we obtain (8.37).

Step 4. We have

$$\int_\Omega A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n) = \int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u) + \omega(n). \tag{8.40}$$

In order to prove (8.40), we use ψ_δ and $(1-\psi_\delta)$. By (8.35), proved in Step 2, we have

$$\int_\Omega A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)\psi_\delta = \omega(n,\delta),$$

while the fact that $T_k(u)$ belongs to $H_0^1(\Omega)$, and the fact that ψ_δ tends to zero in $L^\infty(\Omega)$ weak* implies

$$\int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)\psi_\delta = \omega(\delta).$$

Therefore, in order to prove (8.40) it only remains to prove that

$$\begin{aligned}\int_\Omega & A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)(1-\psi_\delta)\\ &= \int_\Omega A(x)\nabla T_k(u)\cdot\nabla T_k(u)(1-\psi_\delta)+\omega(n,\delta),\end{aligned}$$

which is exactly (8.37).

Step 5. The sequence $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$. Since ∇u_n converges almost everywhere to ∇u, we have that

$$A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n) \text{ tends to } A(x)\nabla T_k(u)\cdot\nabla T_k(u),$$

almost everywhere in Ω. Since $A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)$ is nonnegative, this convergence and (8.40) imply by Lemma 8.5 that

$$A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n) \text{ converges to } A(x)\nabla T_k(u)\cdot\nabla T_k(u),$$

strongly in $L^1(\Omega)$. Therefore, by Vitali theorem,

$$\{A(x)\nabla T_k(u_n)\cdot\nabla T_k(u_n)\} \text{ is equiintegrable.}$$

Using (1.8), this implies that the sequence $\{|\nabla T_k(u_n)|^2\}$ is equiintegrable. Since $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ almost everywhere in Ω, we have (again by Vitali theorem) the strong convergence of $T_k(u_n)$ to $T_k(u)$ in $H_0^1(\Omega)$, as desired. □

THEOREM 8.7. *There exists a renormalized solution of (1.9).*

Proof. Let h and φ as in Definition 8.1, and choose $h(u_n)\,\varphi$ as test function in (8.27). We obtain

$$\begin{aligned}\int_\Omega & A(x)\nabla u_n\cdot\nabla u_n\, h'(u_n)\,\varphi + \int_\Omega A(x)\nabla u_n\cdot\nabla\varphi\, h(u_n)\\ &= \int_\Omega f_n\, h(u_n)\,\varphi + \int_\Omega \lambda_n\, h(u_n)\,\varphi.\end{aligned}$$

For the second and third term we have

$$\int_\Omega A(x)\nabla u_n\cdot\nabla\varphi\, h(u_n) = \int_\Omega A(x)\nabla u\cdot\nabla\varphi\, h(u) + \omega(n),$$

and

$$\int_\Omega f_n\, h(u_n)\,\varphi = \int_\Omega f\, h(u)\,\varphi,$$

since $h(s)$ is bounded, φ is regular, and u_n converges to u strongly in $W_0^{1,1}(\Omega)$ (actually, better) and almost everywhere. For the first one, since $h'(s)$ has compact support (say, $[-M, M]$), we have

$$\int_\Omega A(x)\nabla u_n \cdot \nabla u_n \, h'(u_n)\, \varphi = \int_\Omega A(x)\nabla T_M(u_n) \cdot \nabla T_M(u_n)\, h'(u_n)\, \varphi,$$

and so

$$\int_\Omega A(x)\nabla u_n \cdot \nabla u_n \, h'(u_n)\, \varphi = \int_\Omega A(x)\nabla u \cdot \nabla u \, h'(u)\, \varphi + \omega(n),$$

by the strong convergence of the truncates, the boundedness of $h'(s)$ and the regularity of φ. For the last term we have

$$\begin{aligned}\int_\Omega \lambda_n\, h(u_n)\, \varphi &= \int_\Omega \lambda_n\, (h(u_n) - h^{+\infty})\, \varphi + h^{+\infty} \int_\Omega \lambda_n\, \varphi \\ &= \int_\Omega \lambda_n\, (h(u_n) - h^{+\infty})\, \varphi\, \psi_\delta + \\ &\quad + \int_\Omega \lambda_n\, (h(u_n) - h^{+\infty})\, \varphi\, (1 - \psi_\delta) + h^{+\infty} \int_\Omega \lambda_n\, \varphi.\end{aligned}$$

Since $h'(s)$ has compact support, there exists $K > 0$ such that $|h(s) - h^{+\infty}| \le K - T_K(s)$, so that

$$\left| \int_\Omega \lambda_n\, (h(u_n) - h^{+\infty})\, \varphi\, \psi_\delta \right| \le C \int_\Omega \lambda_n\, (K - T_K(u_n))\, \psi_\delta = \omega(n, \delta),$$

by (8.36). Furthermore,

$$\begin{aligned}\left| \int_\Omega \lambda_n (h(u_n) - h^{+\infty}) \varphi (1 - \psi_\delta) \right| &\le C \int_\Omega \lambda_n (1 - \psi_\delta) \\ &= C \int_\Omega (1 - \psi_\delta) d\lambda + \omega_\delta(n) = \omega(n, \delta),\end{aligned}$$

by (8.26). Therefore,

$$\int_\Omega \lambda_n\, h(u_n)\, \varphi = h^{+\infty} \int_\Omega \varphi\, d\lambda + \omega(n).$$

Putting together all the results, we have that u is a renormalized solution of (1.9). □

We are now going to prove that every renormalized solution is a duality solution, so that uniqueness of renormalized solutions will follow from uniqueness of duality solutions. Before the proof, we need a further result.

THEOREM 8.8. *Let u be a renormalized solution of (1.9). Then*

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{k\le u\le 2k\}} A(x)\nabla u\cdot\nabla u\,\varphi = \int_\Omega \varphi\,d\lambda^+, \tag{8.41}$$

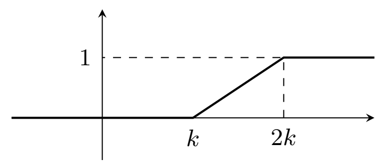
and

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{-2k\le u\le -k\}} A(x)\nabla u\cdot\nabla u\,\varphi = \int_\Omega \varphi\,d\lambda^-, \tag{8.42}$$

for every φ in $C^0(\overline{\Omega})$.

Proof. For the sake of simplicity, we suppose that $\mu_0 = f$, a function in $L^1(\Omega)$. Let $h_k(s):\mathbb{R}\to\mathbb{R}$ be defined by $h_k(s) = \frac{1}{k}T_k(G_k(s^+))$, i.e.,

$$h_k(s) = \begin{cases} 0 & \text{if } s < k, \\ \frac{s-k}{k} & \text{if } k \le s \le 2k, \\ 1 & \text{if } s > 2k, \end{cases}$$



and choose $h_k(u)\,\varphi$ as test function in (8.1), with φ in $C^1(\Omega)$. We have, since $h_k^{+\infty} = 1$, and $h_k^{-\infty} = 0$,

$$\int_\Omega A(x)\nabla u\cdot\nabla(h_k(u)\,\varphi) = \int_\Omega f\,h_k(u)\,\varphi + \int_\Omega \varphi\,d\lambda^+.$$

Since $h_k(u)$ tends to zero almost everyhwere, and is bounded, we have

$$\lim_{k\to+\infty} \int_\Omega f\,h_k(u)\,\varphi = 0,$$

by Lebesgue theorem. For the same reason, we have

$$\lim_{k\to+\infty} \int_\Omega A(x)\nabla u\cdot\nabla\varphi\,h_k(u) = 0,$$

so that

$$\lim_{k\to+\infty} \frac{1}{k} \int_{\{k\le u\le 2k\}} A(x)\nabla u\cdot\nabla u\,\varphi = \int_\Omega \varphi\,d\lambda^+,$$

for every φ in $C^1(\Omega)$. The general result is obtained by a density argument, since the sequence

$$\frac{1}{k}A(x)\nabla u\cdot\nabla u\,\chi_{\{k\le u\le 2k\}}$$

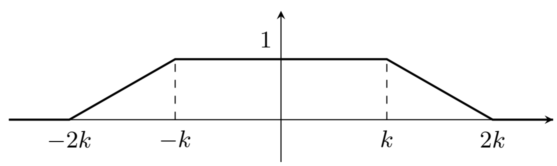
is bounded in $L^1(\Omega)$ (just take $h_k(u)\cdot 1$ as test function in (8.1)). The proof of (8.42) is analogous. $\square$

We can now prove that any renormalized solution is a duality solution. Let g be in $L^\infty(\Omega)$, and let v be the solution of

$$\begin{cases} -\operatorname{div}(A^*(x)\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By De Giorgi's theorem (Theorem 3.4), v belongs to $H_0^1(\Omega) \cap C^0(\overline{\Omega})$, so that, if we define $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(s) = \begin{cases} 0 & \text{if } s < -2k, \\ \frac{s+2k}{k} & \text{if } -2k \le s \le -k, \\ 1 & \text{if } -k < s < k, \\ \frac{2k-s}{k} & \text{if } k \le s \le 2k, \\ 0 & \text{if } s > 2k, \end{cases}$$



we can choose $h_k(u)\,v$ as test function in (8.25) (since $h_k(s)$ has compact support). We obtain

$$\int_\Omega A(x)\nabla u \cdot \nabla v\, h_k(u) + \int_\Omega A(x)\nabla u \cdot \nabla u\, h_k'(u)\, v = \int_\Omega h_k(u) v d\mu_0,$$

since $h_k^{\pm\infty} = 0$. The middle term can be rewritten as

$$\int_\Omega A(x)\nabla u \cdot \nabla u\, h_k'(u)\, v$$
$$= -\frac{1}{k}\int_{\{k<u<2k\}} A(x)\nabla u \cdot \nabla u\, v + \frac{1}{k}\int_{\{-2k<u<-k\}} A(x)\nabla u \cdot \nabla u\, v.$$

so that by (8.41) and (8.42) we have

$$\lim_{k\to+\infty} \int_\Omega A(x)\nabla u \cdot \nabla u\, h_k'(u)\, v = -\int_\Omega v\, d\lambda^+ + \int_\Omega v\, d\lambda^-.$$

Since h_k is bounded and converges to 1 everywhere, the boundedness of v and Lebesgue theorem imply that

$$\lim_{k\to+\infty} \int_\Omega h_k(u) v d\mu_0 = \int_\Omega v\, d\mu_0.$$

For the first term, we can rewrite it as

$$\int_\Omega A(x)\nabla u \cdot \nabla v\, h_k(u) = \int_\Omega A^*(x)\nabla v \cdot \nabla u\, h_k(u) = \int_\Omega A^*(x)\nabla v \cdot \nabla H_k(u),$$

where $H_k(s) = \int_0^s h_k(t)\,dt$. Since h_k has compact support, $H_k(u)$ belongs to $H_0^1(\Omega)$, and so it can be chosen as test function in the equation solved by v, which implies that

$$\int_\Omega A^*(x)\nabla v \cdot \nabla H_k(u) = \int_\Omega g\, H_k(u).$$

Therefore,

$$\int_\Omega A(x)\nabla u \cdot \nabla v\, h_k(u) = \int_\Omega g\, H_k(u).$$

Since $H_k(u)$ tends to u almost everywhere (and is bounded in absolute value by u, which is in $L^1(\Omega)$), by Lebesgue theorem we have

$$\lim_{k\to+\infty} \int_\Omega A(x)\nabla u \cdot \nabla v\, h_k(u) = \int_\Omega g\, u.$$

Putting the results together, we have

$$\int_\Omega g\, u - \int_\Omega v\, d\lambda^+ + \int_\Omega v\, d\lambda^- = \int_\Omega v\, d\mu_0,$$

which can be rewritten as

$$\int_\Omega g\, u = \int_\Omega v\, d\mu,$$

so that u is the duality solution, as desired.

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