A non-local critical problem involving the fractional Laplacian operator

Eduardo Colorado

Universidad Carlos III de Madrid

UGR, Granada Febrero 2012

The main results of the talk are in collaboration with:

- B. Barrios, UAM-ICMAT.
- A. de Pablo, UC3M.
- U. Sánchez, UC3M.

B. Barrios, E. C., A. de Pablo, U. Sánchez, *On Some critical problems for the fractional Laplacian operator.* Preprint 2011, arXiv:1106.6081.

The main problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = f_\lambda(u), \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

The main problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = f_\lambda(u), \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_\lambda(u) = \lambda u^q + u^r$, $\lambda > 0$, $0 < q < r \leq \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$ and $\Omega \subset \mathbb{R}^N$, with $N > \alpha$, $0 < \alpha < 2$.

The main problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = f_\lambda(u), \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_\lambda(u) = \lambda u^q + u^r$, $\lambda > 0$, $0 < q < r \leq \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$ and $\Omega \subset \mathbb{R}^N$, with $N > \alpha$, $0 < \alpha < 2$.

Subcritical case $1 < r < \frac{N+\alpha}{N-\alpha}$, and $q < 1$

[BCdPS] C. Brändle, E.C., A. de Pablo, U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*. To appear in Proc. Roy. Soc. Edinburgh.

Critical case $r = \frac{N+\alpha}{N-\alpha}$

[BCPS] B. Barrios, E.C., A. de Pablo, U. Sánchez, *On Some critical problems for the fractional Laplacian operator.* Preprint 2011, arXiv:1106.6081.

Scheme of the talk

- Definition of the Fractional Laplacian (through the spectral decomposition)

Scheme of the talk

- Definition of the Fractional Laplacian (through the spectral decomposition)
- Extended problem to one more variable

Scheme of the talk

- Definition of the Fractional Laplacian (through the spectral decomposition)
- Extended problem to one more variable
- Trace and Sobolev Inequalities

Scheme of the talk

- Definition of the Fractional Laplacian (through the spectral decomposition)
- Extended problem to one more variable
- Trace and Sobolev Inequalities
- Main results

Scheme of the talk

- Definition of the Fractional Laplacian (through the spectral decomposition)
- Extended problem to one more variable
- Trace and Sobolev Inequalities
- Main results
- Some ideas of the proofs

Definition of the Fractional Laplacian

Powers of Laplacian operator $(-\Delta)$:

Let (λ_n, φ_n) be the eigenvalues and eigenfunctions of $(-\Delta)$ in Ω with zero Dirichlet boundary data. Then $(\lambda_n^{\alpha/2}, \varphi_n)$ are the eigenvalues and eigenfunctions of $(-\Delta)^{\alpha/2}$, also with zero Dirichlet boundary conditions.

Definition of the Fractional Laplacian

Powers of Laplacian operator $(-\Delta)$:

Let (λ_n, φ_n) be the eigenvalues and eigenfunctions of $(-\Delta)$ in Ω with zero Dirichlet boundary data. Then $(\lambda_n^{\alpha/2}, \varphi_n)$ are the eigenvalues and eigenfunctions of $(-\Delta)^{\alpha/2}$, also with zero Dirichlet boundary conditions.

The fractional Laplacian $(-\Delta)^{\alpha/2}$ is well defined in the space

$$H_0^{\alpha/2}(\Omega) = \left\{ u = \sum a_n \varphi_n \in L^2(\Omega) : \ \|u\|^2_{H_0^{\alpha/2}(\Omega)} = \sum a_n^2 \lambda_n^{\alpha/2} < \infty \right\}.$$

Definition of the Fractional Laplacian

Powers of Laplacian operator $(-\Delta)$:

Let (λ_n, φ_n) be the eigenvalues and eigenfunctions of $(-\Delta)$ in Ω with zero Dirichlet boundary data. Then $(\lambda_n^{\alpha/2}, \varphi_n)$ are the eigenvalues and eigenfunctions of $(-\Delta)^{\alpha/2}$, also with zero Dirichlet boundary conditions.

The fractional Laplacian $(-\Delta)^{\alpha/2}$ is well defined in the space

$$H_0^{\alpha/2}(\Omega) = \left\{ u = \sum a_n \varphi_n \in L^2(\Omega) : \ \|u\|^2_{H_0^{\alpha/2}(\Omega)} = \sum a_n^2 \lambda_n^{\alpha/2} < \infty \right\}.$$

As a consequence,

$$(-\Delta)^{\alpha/2} u = \sum \lambda_n^{\alpha/2} a_n \varphi_n.$$

Note that then $\|u\|_{H_0^{\alpha/2}(\Omega)} = \|(-\Delta)^{\alpha/4} u\|_{L^2(\Omega)}$.

Definition of the Fractional Laplacian

We now consider the general problem

$$(P) \quad \begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition of the Fractional Laplacian

We now consider the general problem

$$(P) \quad \begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that $u \in H_0^{\alpha/2}(\Omega)$ is an energy solution of (P) if the identity

$$\int_\Omega (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi \, dx = \int_\Omega f(x,u) \varphi \, dx$$

holds for $\forall\, \varphi \in H_0^{\alpha/2}(\Omega)$.

Definition of the Fractional Laplacian

$$
(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\lambda > 0$, $0 < q < \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$ and $\Omega \subset \mathbb{R}^N$, with $N > \alpha$, $0 < \alpha < 2$.

Definition of the Fractional Laplacian

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $0 < q < \frac{N+\alpha}{N-\alpha} = 2_\alpha^* - 1$ and $\Omega \subset \mathbb{R}^N$, with $N > \alpha$, $0 < \alpha < 2$.

By the definition of solution, if $f_\lambda(u) = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}$

$$\int_\Omega (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi \, dx = \int_\Omega f_\lambda(u) \varphi \, dx, \quad \forall \varphi \in H_0^{\alpha/2}(\Omega).$$

Definition of the Fractional Laplacian

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, $0 < q < \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$ and $\Omega \subset \mathbb{R}^N$, with $N > \alpha$, $0 < \alpha < 2$.

By the definition of solution, if $f_\lambda(u) = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}$

$$\int_\Omega (-\Delta)^{\alpha/4}u(-\Delta)^{\alpha/4}\varphi\, dx = \int_\Omega f_\lambda(u)\varphi\, dx, \quad \forall\varphi \in H_0^{\alpha/2}(\Omega).$$

Since $u \in H_0^{\alpha/2}(\Omega) \Rightarrow f(u) \in L^{\frac{2N}{N+\alpha}}(\Omega) \hookrightarrow H^{-\alpha/2}(\Omega)$.

Then $f_\lambda(u)\varphi \in L^1(\Omega)$.

Associated energy functional

$$I(u) = \frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/4}u\right|^2 dx - \frac{\lambda}{q+1}\int_\Omega u^{q+1}\, dx - \frac{1}{2^*_\alpha}\int_\Omega u^{2^*_\alpha} dx$$

which is well defined in $H_0^{\alpha/2}(\Omega)$. Clearly, the critical points of I correspond to solutions to (P_λ).

Extended problems to one more variable

Consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. Given $u \in H_0^{\alpha/2}(\Omega)$, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} -\text{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Extended problems to one more variable

Consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. Given $u \in H_0^{\alpha/2}(\Omega)$, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} -\mathrm{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

The extension function belongs to the space $X_0^\alpha(\mathcal{C}_\Omega)$ defined as the completion of $\{z \in \mathcal{C}^\infty(\mathcal{C}_\Omega) : \ z = 0 \text{ on } \partial_L \mathcal{C}_\Omega\}$ with the norm

$$\|z\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \left(\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z|^2 dx dy \right)^{1/2}$$

where κ_α is a normalization constant.

Extended problems to one more variable

Consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$. Given $u \in H_0^{\alpha/2}(\Omega)$, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} -\text{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0, \infty), \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

With that κ_α, the extension operator is an isometry

$$\| \mathrm{E}_\alpha(\psi) \|_{X_0^\alpha(\mathcal{C}_\Omega)} = \|\psi\|_{H_0^{\alpha/2}(\Omega)}, \quad \forall\, \psi \in H_0^{\alpha/2}(\Omega).$$

Extended problems to one more variable

Consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. Given $u \in H_0^{\alpha/2}(\Omega)$, we define its α-harmonic extension $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} -\text{div}(y^{1-\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

With that κ_α, the extension operator is an isometry

$$\| \mathrm{E}_\alpha(\psi)\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \|\psi\|_{H_0^{\alpha/2}(\Omega)}, \quad \forall\, \psi \in H_0^{\alpha/2}(\Omega).$$

Moreover, for any $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$, we have the following trace inequality

$$\|\varphi\|_{X_0^\alpha(\mathcal{C}_\Omega)} \geq \|\varphi(\cdot,0)\|_{H_0^{\alpha/2}(\Omega)}.$$

Extended problems to one more variable

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$-\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y) = (-\Delta)^{\alpha/2}u(x),$$

Extended problems to one more variable

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$-\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y) = (-\Delta)^{\alpha/2}u(x),$$

See:

[CS] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations, 2007.

See also:

[BCdPS] C. Brändle, E.C., A. de Pablo, U. Sánchez, To appear in Proc. Roy. Soc. Edinburgh.

[CT] X. Cabré, J. Tan, Adv. Math., 2010.

[CDDS] A. Capella, J. Dávila, L. Dupaigne, Y. Sire, To appear in Comm. Partial Differential Equations.

Extended problems to one more variable

When $\Omega = \mathbb{R}^N$, the above Dirichlet to Neumann procedure provides a formula to the fractional Laplacian in the whole space equivalent to the one by Fourier Transform,

$$((-\Delta)^{\alpha/2})g)\hat{}(\xi) = |\xi|^\alpha \hat{g}(\xi).$$

See

[CS] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations, 2007.

Extended problems to one more variable

When $\Omega = \mathbb{R}^N$, the above Dirichlet to Neumann procedure provides a formula to the fractional Laplacian in the whole space equivalent to the one by Fourier Transform,

$$((-\Delta)^{\alpha/2})g)\hat{}(\xi) = |\xi|^\alpha \hat{g}(\xi).$$

See

[CS] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations, 2007.

In that case there are explicit expressions to the α-harmonic extension and the fractional Laplacian in terms of the Poisson and Riesz kernels, resp.

$$w(x,y) = P_y^\alpha * u(x) = c_{N,\alpha} y^\alpha \int_{\mathbb{R}^N} \frac{u(s)}{(|x-s|^2 + y^2)^{\frac{N+\alpha}{2}}}\, ds,$$

$$(-\Delta)^{\alpha/2} u(x) = d_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(s)}{|x-s|^{N+\alpha}}\, ds.$$

$\alpha\, c_{N,\alpha} \kappa_\alpha = d_{N,\alpha}$

Extended problems to one more variable

Denoting

$$L_\alpha w := -\mathrm{div}(y^{1-\alpha}\nabla w), \qquad \frac{\partial w}{\partial \nu^\alpha} := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}$$

we can reformulate (P_λ) with the new variable as

$$(\overline{P}_\lambda) \quad \begin{cases} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega \times \{y = 0\}. \end{cases}$$

Extended problems to one more variable

Denoting

$$L_\alpha w := -\mathrm{div}(y^{1-\alpha}\nabla w), \qquad \frac{\partial w}{\partial \nu^\alpha} := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}$$

we can reformulate (P_λ) with the new variable as

$$(\overline{P}_\lambda) \quad \begin{cases} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega \times \{y = 0\}. \end{cases}$$

$w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution if

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla \varphi\rangle \, dxdy = \int_\Omega \left(\lambda w^q + w^{\frac{N+\alpha}{N-\alpha}}\right)\varphi \, dx, \qquad \forall\, \varphi \in X_0^\alpha(\mathcal{C}_\Omega).$$

Extended problems to one more variable

Denoting

$$L_\alpha w := -\mathrm{div}(y^{1-\alpha}\nabla w), \qquad \frac{\partial w}{\partial \nu^\alpha} := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}$$

we can reformulate (P_λ) with the new variable as

$$(\overline{P}_\lambda) \quad \begin{cases} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega\times\{y=0\}. \end{cases}$$

$w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution if

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle\, dxdy = \int_\Omega \left(\lambda w^q + w^{\frac{N+\alpha}{N-\alpha}}\right)\varphi\, dx, \qquad \forall\, \varphi \in X_0^\alpha(\mathcal{C}_\Omega).$$

Energy functional:

$$J(w) = \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \frac{\lambda}{q+1}\int_\Omega w^{q+1}\, dx - \frac{1}{2^*_\alpha}\int_\Omega w^{2^*_\alpha}\, dx\,.$$

Extended problems to one more variable

Denoting

$$L_\alpha w := -\mathrm{div}(y^{1-\alpha}\nabla w), \qquad \frac{\partial w}{\partial \nu^\alpha} := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}$$

we can reformulate (P_λ) with the new variable as

$$(\overline{P}_\lambda) \quad \begin{cases} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega \times \{y = 0\}. \end{cases}$$

$w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution if

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla \varphi\rangle \, dxdy = \int_\Omega \left(\lambda w^q + w^{\frac{N+\alpha}{N-\alpha}}\right)\varphi \, dx, \qquad \forall\, \varphi \in X_0^\alpha(\mathcal{C}_\Omega).$$

Energy functional:

$$J(w) = \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 \, dxdy - \frac{\lambda}{q+1}\int_\Omega w^{q+1}\, dx - \frac{1}{2^*_\alpha}\int_\Omega w^{2^*_\alpha}\, dx\,.$$

Note that critical points of J in $X_0^\alpha(\mathcal{C}_\Omega)$ correspond to critical points of I in $H_0^{\alpha/2}(\Omega)$. Even more, minima of J also correspond to minima of I.

Sobolev and Trace inequalities

Assume $N > \alpha$, there exists a positive constant $C = C(\alpha, r, N, \Omega)$ such that for $1 \leq r \leq 2_\alpha^* = \frac{2N}{N-\alpha}$,

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy \geq C \left(\int_\Omega |z(x,0)|^r \, dx \right)^{2/r}$$

for any $z \in X_0^\alpha(\mathcal{C}_\Omega)$.

Sobolev and Trace inequalities

Assume $N > \alpha$, there exists a positive constant $C = C(\alpha, r, N, \Omega)$ such that for $1 \le r \le 2_\alpha^* = \frac{2N}{N-\alpha}$,

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy \ge C \left(\int_\Omega |z(x,0)|^r \, dx \right)^{2/r}$$

for any $z \in X_0^\alpha(\mathcal{C}_\Omega)$.

Also,

$$\int_\Omega |(-\Delta)^{\alpha/4} v|^2 \, dx \ge C \left(\int_\Omega |v|^r \, dx \right)^{2/r}$$

for any $v \in H_0^{\alpha/2}(\Omega)$.

Sobolev and Trace inequalities

When $\Omega = \mathbb{R}^N$, $r = 2_\alpha^*$, there exists a constant $S(\alpha, N) > 0$ such that

$$\int_{\mathbb{R}_+^{N+1}} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy \geq S(\alpha, N) \left(\int_{\mathbb{R}^N} |z(x,0)|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*}, \quad \forall \, z \in X^\alpha(\mathbb{R}_+^{N+1}).$$

The constant is achieved when $z(\cdot, 0) = u(\cdot)$ takes the form:

$$u(x) = u_\varepsilon(x) = \frac{\varepsilon^{(N-\alpha)/2}}{(|x|^2 + \varepsilon^2)^{(N-\alpha)/2}}.$$

Sobolev and Trace inequalities

When $\Omega = \mathbb{R}^N$, $r = 2^*_\alpha$, there exists a constant $S(\alpha, N) > 0$ such that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy \geq S(\alpha, N) \left(\int_{\mathbb{R}^N} |z(x,0)|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}, \quad \forall\, z \in X^\alpha(\mathbb{R}^{N+1}_+).$$

The constant is achieved when $z(\cdot, 0) = u(\cdot)$ takes the form:

$$u(x) = u_\varepsilon(x) = \frac{\varepsilon^{(N-\alpha)/2}}{(|x|^2 + \varepsilon^2)^{(N-\alpha)/2}}.$$

Also we have the corresponding Sobolev inequality

$$\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4} v|^2 \, dx \geq \kappa_\alpha S(\alpha, N) \left(\int_{\mathbb{R}^N} |v|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}$$

for any $v \in H^{\alpha/2}_0(\mathbb{R}^N)$.

Sobolev and Trace inequalities

When $\Omega = \mathbb{R}^N$, $r = 2^*_\alpha$, there exists a constant $S(\alpha, N) > 0$ such that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla z(x,y)|^2 \, dxdy \geq S(\alpha, N) \left(\int_{\mathbb{R}^N} |z(x,0)|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}, \quad \forall\, z \in X^\alpha(\mathbb{R}^{N+1}_+).$$

The constant is achieved when $z(\cdot, 0) = u(\cdot)$ takes the form:

$$u(x) = u_\varepsilon(x) = \frac{\varepsilon^{(N-\alpha)/2}}{(|x|^2 + \varepsilon^2)^{(N-\alpha)/2}}.$$

Also we have the corresponding Sobolev inequality

$$\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4} v|^2 \, dx \geq \kappa_\alpha S(\alpha, N) \left(\int_{\mathbb{R}^N} |v|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}$$

for any $v \in H_0^{\alpha/2}(\mathbb{R}^N)$.

- Note that these constants are achived on $\mathbb{R}^N$, but are not attained in any bounded domain.

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. There exists $0 < \Lambda < \infty$ such that the problem (P_λ)

1. has no solution for $\lambda > \Lambda$;
2. has at least two solutions for each $0 < \lambda < \Lambda$; ($1 \leq \alpha < 2$)
3. has a solution for $\lambda = \Lambda$.

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. There exists $0 < \Lambda < \infty$ such that the problem (P_λ)

1. has no solution for $\lambda > \Lambda$;
2. has at least two solutions for each $0 < \lambda < \Lambda$; ($1 \leq \alpha < 2$)
3. has a solution for $\lambda = \Lambda$.

In the subcritical case **[BCdPS]** the same restriction on α appeared. The difficulty was to find a Liouville-type theorem for $0 < \alpha < 1$. Here, due to the lack of regularity, it is not clear how to separate the solutions in the appropriate way, see **[CP,D]** for more details.

[BCdPS] C. Brändle, E.C., A. de Pablo, U. Sánchez, To appear in Proc. Roy. Soc. Edinburgh.
[CP] E. C., I. Peral J. Funct. Anal. 2003.
[D] J. Dávila, J. Funct. Anal. 2001.

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. There exists $0 < \Lambda < \infty$ such that the problem (P_λ)

1. has no solution for $\lambda > \Lambda$;
2. has at least two solutions for each $0 < \lambda < \Lambda$; ($1 \leq \alpha < 2$)
3. has a solution for $\lambda = \Lambda$.

Theorem 2 Let $q = 1$, $0 < \alpha < 2$ and $N \geq 2\alpha$. Then the problem (P_λ)

1. has no solution for $\lambda \geq \lambda_1$;
2. has a solution for each $0 < \lambda < \lambda_1$.

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. There exists $0 < \Lambda < \infty$ such that the problem (P_λ)

1. has no solution for $\lambda > \Lambda$;
2. has at least two solutions for each $0 < \lambda < \Lambda$; ($1 \leq \alpha < 2$)
3. has a solution for $\lambda = \Lambda$.

Theorem 2 Let $q = 1$, $0 < \alpha < 2$ and $N \geq 2\alpha$. Then the problem (P_λ)

1. has no solution for $\lambda \geq \lambda_1$;
2. has a solution for each $0 < \lambda < \lambda_1$.

We have left open the range $\alpha < N < 2\alpha$. See the special case $\alpha = 2$ and $N = 3$ in **[BN]**. If $\alpha = 1$ this range is empty, see **[T]**.

[BN] H. Brezis, L. Nirenberg, Comm. Pure Appl. Math. 1983.
[T] J. Tan, Calc. Var. Partial Differential Equations 2011.

Main Results

Remember the problem

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = \lambda u^q + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. There exists $0 < \Lambda < \infty$ such that the problem (P_λ)

1. has no solution for $\lambda > \Lambda$;
2. has at least two solutions for each $0 < \lambda < \Lambda$; ($1 \leq \alpha < 2$)
3. has a solution for $\lambda = \Lambda$.

Theorem 2 Let $q = 1$, $0 < \alpha < 2$ and $N \geq 2\alpha$. Then the problem (P_λ)

1. has no solution for $\lambda \geq \lambda_1$;
2. has a solution for each $0 < \lambda < \lambda_1$.

Theorem 3 Let $1 < q < 2^*_\alpha - 1$, $0 < \alpha < 2$ and $N > \alpha(1 + (1/q))$. Then the problem (P_λ) has a solution for any $\lambda > 0$.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \le f(x,s) \le C(1+|s|^p) \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \le 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \leq f(x,s) \leq C(1+|s|^p) \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \leq 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

The proof follows by the Moser iterative method (**[GT]**) with appropriate test functions.

[GT] D. Gilbarg, N.S. Trudinger, "Elliptic partial differential equations of second order" Springer-Verlag, Berlin 2001.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \le f(x,s) \le C(1+|s|^p) \quad \forall\, (x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \le 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proposition 2 Let u be a solution of (P_λ).

(i) If $\alpha = 1$ and $q \ge 1$ then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

(ii) If $\alpha = 1$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $\alpha < 1$ then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

(iv) If $\alpha > 1$ then $u \in \mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \le f(x,s) \le C(1+|s|^p) \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \le 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proposition 2 Let u be a solution of (P_λ).

(i) If $\alpha = 1$ and $q \ge 1$ then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

(ii) If $\alpha = 1$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $\alpha < 1$ then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

(iv) If $\alpha > 1$ then $u \in \mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.

Proof: (i) By Proposition 1 and **[CT]** we get that $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$, for some $\gamma < 1$. Since $q \ge 1$ then $f_\lambda(u) \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$. Again by **[CT]**, it follows that $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$. Iterating the process we conclude that $u \in \mathcal{C}^\infty(\overline{\Omega})$.

[CT] X. Cabré, J. Tan, Adv. Math. 2010.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \leq f(x,s) \leq C(1+|s|^p) \quad \forall\, (x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \leq 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proposition 2 Let u be a solution of (P_λ).

(i) If $\alpha = 1$ and $q \geq 1$ then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

(ii) If $\alpha = 1$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $\alpha < 1$ then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

(iv) If $\alpha > 1$ then $u \in \mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.

Proof: (ii) As before we have $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$, for some $\gamma < 1$. Therefore $f_\lambda(u) \in \mathcal{C}^{0,q\gamma}(\overline{\Omega})$. It follows that $u \in \mathcal{C}^{1,q\gamma}(\overline{\Omega})$, which gives $f_\lambda(u) \in \mathcal{C}^{0,q}(\overline{\Omega})$. Finally this implies $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \le f(x,s) \le C(1+|s|^p) \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \le 2_\alpha^* - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proposition 2 Let u be a solution of (P_λ).

(i) If $\alpha = 1$ and $q \ge 1$ then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

(ii) If $\alpha = 1$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $\alpha < 1$ then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

(iv) If $\alpha > 1$ then $u \in \mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.

Proof: (iii) By **[CDDS]** we obtain that $u \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$ for all $\gamma \in (0,\alpha)$. This implies that $f_\lambda(u) \in \mathcal{C}^{0,r}(\overline{\Omega})$ for every $r < \min\{q\alpha,\ \alpha\}$. Therefore, again by another result in **[CDDS]**, we get that $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

[CDDS] A. Capella, J. Dávila, L. Dupaigne, Y. Sire, To appear in Comm. Partial Differential Equations, arXiv:1004.1906.

Auxiliary results (regularity)

Proposition 1 Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \le f(x,s) \le C(1+|s|^p) \quad \forall\, (x,s) \in \Omega \times \mathbb{R},$ and some $0 < p \le 2^*_\alpha - 1$. Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proposition 2 Let u be a solution of (P_λ).

(i) If $\alpha = 1$ and $q \ge 1$ then $u \in \mathcal{C}^\infty(\overline{\Omega})$.

(ii) If $\alpha = 1$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $\alpha < 1$ then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

(iv) If $\alpha > 1$ then $u \in \mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.

Proof: (iv) Since $\alpha > 1$, we can write problem (P_λ) as follows

$$\begin{cases} (-\Delta)^{1/2} u = s & \text{in } \Omega, \\ (-\Delta)^{(\alpha-1)/2} s = f_\lambda(u) & \text{in } \Omega, \\ u = s = 0 & \text{on } \partial\Omega. \end{cases}$$

Reasoning as before, we obtain the desired regularity in two steps, using **[CT]** and **[CDDS]**.

Auxiliary results (concentration-compactness)

Following the classical result by P. L. Lions in **[L]**.

[L] P. L. Lions Rev. Mat. Iberoamericana Part II, 1985.

Auxiliary results (concentration-compactness)

Following the classical result by P. L. Lions in **[L]**.

[L] P. L. Lions Rev. Mat. Iberoamericana Part II, 1985.

Proposition 3 Let $\{w_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence to w in $X_0^\alpha(\mathcal{C}_\Omega)$, such that the sequence $\left\{y^{1-\alpha}|\nabla w_n|^2\right\}_{n\in\mathbb{N}}$ is tight. Let $u_n = Tr(w_n)$ and $u = Tr(w)$. Assume that μ, ν are two non negative measures such that

$$y^{1-\alpha}|\nabla w_n|^2 \to \mu \qquad \text{and} \qquad |u_n|^{2^*_\alpha} \to \nu, \quad \text{as } n \to \infty \tag{0.2}$$

in the sense of measures. Then there exist an at most countable set I, points $\{x_k\}_{k\in I} \subset \Omega$ and real positive numbers μ_k, ν_k such that

1. $\mu \geq y^{1-\alpha}|\nabla w|^2 + \sum_{k\in I} \mu_k \delta_{x_k}$,
2. $\nu = |u|^{2^*_\alpha} + \sum_{k\in I} \nu_k \delta_{x_k}$,
3. $\mu_k \geq S(\alpha, N)\nu_k^{\frac{2}{2^*_\alpha}}$.

Main ideas/steps of the proofs

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. Then, there exists $0 < \Lambda < \infty$ such that Problem (P_λ)

1. has no positive solution for $\lambda > \Lambda$;
2. has at least two positive solutions for each $0 < \lambda < \Lambda$;
3. has a positive solution for $\lambda = \Lambda$.

Main ideas/steps of the proofs

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. Then, there exists $0 < \Lambda < \infty$ such that Problem (P_λ)

1. has no positive solution for $\lambda > \Lambda$;
2. has at least two positive solutions for each $0 < \lambda < \Lambda$;
3. has a positive solution for $\lambda = \Lambda$.

Proof of Theorem 1 1. Denoting (λ_1, φ_1) the first eigenvalue and an associated positive eigenfunction to the classical Laplace operator, we have that

$$\int_\Omega \left(\lambda u^q + u^{\frac{N+\alpha}{N-\alpha}} \right) \varphi_1 \, dx = \lambda_1^{\alpha/2} \int_\Omega u \varphi_1 \, dx.$$

Observe that there exist positive constants c, δ such that $\lambda t^q + t^{\frac{N+\alpha}{N-\alpha}} > c\lambda^\delta t$, for any $t > 0$, hence by the previous integral identity, $c\lambda^\delta < \lambda_1^{\alpha/2} \Rightarrow \Lambda < \infty$.

Main ideas/steps of the proofs

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. Then, there exists $0 < \Lambda < \infty$ such that Problem (P_λ)

1. has no positive solution for $\lambda > \Lambda$;
2. has at least two positive solutions for each $0 < \lambda < \Lambda$;
3. has a positive solution for $\lambda = \Lambda$.

Proof of Theorem 1 To prove $\Lambda > 0$, for $\lambda > 0$ sufficiently small, one can use the iteration method of sub-supersolutions, starting with the subsolution and obtaining a minimal one.

See for example, the pioneering works **[GP,ABC]** for the p-Laplacian, Laplacian resp. among others...

Moreover, it is easy to see that we have an interval of minimal solutions increasing with respect to λ for any $0 < \lambda < \Lambda$.

[GP] J. García-Azorero, I. Peral Trans. Amer. Math. Soc. 1991.

[ABC] A. Ambrosetti, H. Brezis, G. Cerami J. Func. Analysis 1994.

Main ideas/steps of the proofs

Theorem 1 Let $0 < q < 1$, $1 \le \alpha < 2$. Then, there exists $0 < \Lambda < \infty$ such that Problem (P_λ)

1. has no positive solution for $\lambda > \Lambda$;
2. has at least two positive solutions for each $0 < \lambda < \Lambda$;
3. has a positive solution for $\lambda = \Lambda$.

Proof of Theorem 1 3. The idea consist (like in **[ABC]**) on passing to the limit as $\lambda \nearrow \Lambda$ on the sequence of minimal solutions $w_n = w_{\lambda_n}$. Clearly $J_{\lambda_n}(w_n) < 0$, hence

$$0 > J_{\lambda_n}(w_n) - \frac{1}{2^*_\alpha}\langle J'_{\lambda_n}(w_n), w_n\rangle = \kappa_\alpha \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right) \|w_n\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda_n \left(\frac{1}{q+1} - \frac{1}{2^*_\alpha}\right) \int_\Omega w_n^{q+1} dx.$$

By the Sobolev and Trace inequalities, the sequence is bounded, $\|w_n\|_{X_0^\alpha} \le C$. Then there exist a subsequence $w_n \rightharpoonup w_\Lambda \in X_0^\alpha(\mathcal{C}_\Omega)$. Moreover, by comparison $w_\Lambda \ge w_\lambda > 0$ for any $0 < \lambda < \Lambda$. A non trivial solution to (P_Λ).

Main ideas/steps of the proofs

Theorem 1 Let $0 < q < 1$, $1 \leq \alpha < 2$. Then, there exists $0 < \Lambda < \infty$ such that Problem (P_λ)

1. has no positive solution for $\lambda > \Lambda$;
2. has at least two positive solutions for each $0 < \lambda < \Lambda$;
3. has a positive solution for $\lambda = \Lambda$.

Proof of Theorem 1 2. Ideas/Steps:

(i) The minimal solution is a local minimum for the functional.

(ii) So we can use the Mountain Pass Theorem, obtaining a minimax sequence.

(iii) In order to find a second solution, we need to prove a Palais-Smale (PS) condition under a critical level.

(iv) Arguing by contradiction, if the local minimum would be the unique critical point, then the functional satisfies a local $(PS)_c$ condition for c under a critical level. To do that we construct a path by localizing the minimizers of the Trace/Sobolev inequalities at the possible Dirac Deltas given by the concentration-compactness result.

Here appear new difficulties:

1.- It is not known how the fractional Laplacian acts on products of functions.

2.- By that, we work on the extended functional, but the minimizers have not an explicit expression on $\mathbb{R}^{N+1}_+$.

3.- Also we need to prove that there is neither vanishing, nor dichotomy, since we pass to the infinite cylinder in the y-variable.

Main ideas/steps of the proofs

In order to prove that the minimal solution is a local minimum, first we show that is a local minimum in the $\mathcal{C}^1$-topology. To start with we prove a separation lemma in that topology.

Main ideas/steps of the proofs

In order to prove that the minimal solution is a local minimum, first we show that is a local minimum in the $\mathcal{C}^1$-topology. To start with we prove a separation lemma in that topology.

Lemma 1 Let $0 < \mu_1 < \lambda_0 < \mu_2 < \Lambda$. Let z_{μ_1}, z_{λ_0} and z_{μ_2} be the corresponding minimal solutions to (P_λ), $\lambda = \mu_1,\ \lambda_0$ and μ_2 respectively. If $\mathcal{X} = \{z \in \mathcal{C}^1_0(\Omega) |\, z_{\mu_1} \leq z \leq z_{\mu_2}\}$, then there exists $\varepsilon > 0$ such that

$$\{z_{\lambda_0}\} + \varepsilon B_1 \subset \mathcal{X},$$

where B_1 is the unit ball in $\mathcal{C}^1_0(\Omega)$.

Main ideas/steps of the proofs

In order to prove that the minimal solution is a local minimum, first we show that is a local minimum in the $\mathcal{C}^1$-topology. To start with we prove a separation lemma in that topology.

Lemma 1 Let $0 < \mu_1 < \lambda_0 < \mu_2 < \Lambda$. Let z_{μ_1}, z_{λ_0} and z_{μ_2} be the corresponding minimal solutions to (P_λ), $\lambda = \mu_1,\ \lambda_0$ and μ_2 respectively. If $\mathcal{X} = \{z \in \mathcal{C}^1_0(\Omega) |\, z_{\mu_1} \le z \le z_{\mu_2}\}$, then there exists $\varepsilon > 0$ such that

$$\{z_{\lambda_0}\} + \varepsilon B_1 \subset \mathcal{X},$$

where B_1 is the unit ball in $\mathcal{C}^1_0(\Omega)$.

Proof. Since $\alpha \ge 1$, by Proposition 2, $\exists\, 0 < \gamma < 1$ such that any solution to (P_λ) is in $\mathcal{C}^{1,\gamma}$ for any $0 < \lambda < \Lambda$. Then

$$u(x) \le C\, \text{dist}\,(x, \partial\Omega), \quad \forall\, x \in \Omega.$$

By comparison with a positive first eigenfunction of the Laplacian, we get

$$u(x) \ge c\, \text{dist}\,(x, \partial\Omega), \quad \forall\, x \in \Omega. \quad \Box$$

Main ideas/steps of the proofs

Lemma 2 For all $\lambda \in (0, \Lambda)$ there exists a solution for (P_λ) which is a local minimum of the functional I in the $\mathcal{C}^1$-topology.

Main ideas/steps of the proofs

Lemma 2 For all $\lambda \in (0, \Lambda)$ there exists a solution for (P_λ) which is a local minimum of the functional I in the $\mathcal{C}^1$-topology.

Proof. Given $0 < \mu_1 < \lambda < \mu_2 < \Lambda$, let z_{μ_1} and z_{μ_2} be the minimal solutions of (P_{μ_1}) and (P_{μ_2}) respectively, we set

$$f^*(x,s) = \begin{cases} f_\lambda(z_{\mu_1}(x)) & \text{if } s \le z_{\mu_1}, \\ f_\lambda(s) & \text{if } z_{\mu_1} \le s \le z_{\mu_2}, \\ f_\lambda(z_{\mu_2}(x)) & \text{if } z_{\mu_2} \le s, \end{cases}$$

$$F^*(x,z) = \int_0^z f^*(x,s)\,ds$$

and

$$I^*(z) = \frac{1}{2}\|z\|_{H_0^{\alpha/2}(\Omega)} - \int_\Omega F^*(x,u)dx.$$

This functional achieves its global minimum.

By comparison with our functional in $\mathcal{X}$ and Lemma 1 we get a minimum in $\mathcal{C}_0^1(\Omega)$.

Main ideas/steps of the proofs

We check that the theorem in **[BN2]** is easy to prove in our setting.

Proposition 3 Let $z_0 \in H_0^{\alpha/2}(\Omega)$ be a local minimum of I in $\mathcal{C}_0^1(\Omega)$, i.e., there exists $r > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in \mathcal{C}_0^1(\Omega) \text{ with } \|z\|_{\mathcal{C}_0^1(\Omega)} \leq r. \tag{0.3}$$

Then z_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in H_0^{\alpha/2}(\Omega) \text{ with } \|z\|_{H_0^{\alpha/2}(\Omega)} \leq \varepsilon_0.$$

Main ideas/steps of the proofs

We check that the theorem in **[BN2]** is easy to prove in our setting.

Proposition 3 Let $z_0 \in H_0^{\alpha/2}(\Omega)$ be a local minimum of I in $\mathcal{C}_0^1(\Omega)$, i.e., there exists $r > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in \mathcal{C}_0^1(\Omega) \text{ with } \|z\|_{\mathcal{C}_0^1(\Omega)} \leq r. \tag{0.4}$$

Then z_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in H_0^{\alpha/2}(\Omega) \text{ with } \|z\|_{H_0^{\alpha/2}(\Omega)} \leq \varepsilon_0.$$

[BN2] H. Brezis, L. Nirenberg H^1 *versus* $\mathcal{C}^1$ *local minimizers* CRAS 1993.

Main ideas/steps of the proofs

We check that the theorem in **[BN2]** is easy to prove in our setting.

Proposition 3 Let $z_0 \in H_0^{\alpha/2}(\Omega)$ be a local minimum of I in $\mathcal{C}_0^1(\Omega)$, i.e., there exists $r > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in \mathcal{C}_0^1(\Omega) \text{ with } \|z\|_{\mathcal{C}_0^1(\Omega)} \leq r. \tag{0.5}$$

Then z_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$I(z_0) \leq I(z_0 + z) \qquad \forall z \in H_0^{\alpha/2}(\Omega) \text{ with } \|z\|_{H_0^{\alpha/2}(\Omega)} \leq \varepsilon_0.$$

We make a translation in the nonlinearity of the functional in order to get that minimum at the origin.

Lemma 3 (i) The translated functional has a local minimum at the origin in $H_0^{\alpha/2}(\Omega)$. Moreover, (ii) the extended functional has a local minimum at the origin in $X_0^{\alpha}(\mathcal{C}_\Omega)$.

Main ideas/steps of the proofs

We check that the theorem in **[BN2]** is easy to prove in our setting.

Proposition 3 Let $z_0 \in H_0^{\alpha/2}(\Omega)$ be a local minimum of I in $\mathcal{C}_0^1(\Omega)$, i.e., there exists $r > 0$ such that

$$I(z_0) \le I(z_0 + z) \qquad \forall z \in \mathcal{C}_0^1(\Omega) \text{ with } \|z\|_{\mathcal{C}_0^1(\Omega)} \le r. \tag{0.6}$$

Then z_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$I(z_0) \le I(z_0 + z) \qquad \forall z \in H_0^{\alpha/2}(\Omega) \text{ with } \|z\|_{H_0^{\alpha/2}(\Omega)} \le \varepsilon_0.$$

We make a translation in the nonlinearity of the functional in order to get that minimum at the origin.

Lemma 3 (i) The translated functional has a local minimum at the origin in $H_0^{\alpha/2}(\Omega)$. Moreover, (ii) the extended functional has a local minimum at the origin in $X_0^{\alpha}(\mathcal{C}_\Omega)$.

The part (i) follows by simple computations as in the classical case, **[ABC]**. The second part (ii) follows by using the isometry between $H_0^{\alpha/2}(\Omega)$ and $X_0^{\alpha}(\mathcal{C}_\Omega)$, and the fact that the α-harmonic extension minimize the norm in $X_0^{\alpha}(\mathcal{C}_\Omega)$.

[ABC] A. Ambrosetti, H. Brezis, G. Cerami J. Funct. Analysis 1994.

Main ideas/steps of the proofs

Lemma 4 If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local (PS)$_c$ condition for any $c < c^*$.

Main ideas/steps of the proofs

Lemma 4 If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local (PS)$_c$ condition for any $c < c^*$.

In order to prove it, first we show that the (PS)$_c$ sequence of the previous lemma is tight.

Lemma 5 For any $\eta > 0$ there exists $\rho_0 > 0$ such that

$$\int_{\{y>\rho_0\}} \int_\Omega y^{1-\alpha} |\nabla z_n|^2 dxdy < \eta, \quad \forall\, n \in \mathbb{N}.$$

Main ideas/steps of the proofs

Lemma 4 If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local (PS)$_c$ condition for any $c < c^*$.

Proof of Lemma 4. Let $\{w_n\}$ be a $(PS)_c$: $\widetilde{J}(w_n) \to c < c^*, \quad \widetilde{J}'(w_n) \to 0$. By Lemma 5 and Proposition 3 (concentration-compactness), there exists an index set I (at most countable) and a sequence of points $\{x_k\} \subset \Omega$, $k \in I$ and real positive numbers μ_k, ν_k such that (up to a subsequence)

$$y^{1-\alpha}|\nabla w_n|^2 \to \mu \geq y^{1-\alpha}|\nabla w_0|^2 + \sum_{k\in I} \mu_k \delta_{x_k}$$

and

$$|w_n(\cdot,0)|^{2^*_\alpha} \to \nu = |w_0(\cdot,0)|^{2^*_\alpha} + \sum_{k\in I} \nu_k \delta_{x_k}$$

in the sense of measures, and moreover, $\mu_k \geq S(\alpha,N)\nu_k^{\frac{2}{2^*_\alpha}}$, for every $k \in I$.

Main ideas/steps of the proofs

Lemma 4 If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local (PS)$_c$ condition for any $c < c^*$.

Proof of Lemma 4. Let ϕ be a regular non-increasing cut-off function, $\phi = 1$ in B_1, $\phi = 0$ in B_2^c. Then $\phi_\varepsilon(x,y) = \phi(x/\varepsilon, y/\varepsilon)$, it is clear that $|\nabla\phi_\varepsilon| \leq \frac{C}{\varepsilon}$. We denote
$\Gamma_{2\varepsilon} = B_{2\varepsilon}^+(x_{k_0}) \cap \{y = 0\}$.
Clearly,

$$\kappa_\alpha \lim_{n\to\infty} \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla w_n, \nabla \phi_\varepsilon \rangle w_n dxdy$$

$$= \lim_{n\to\infty} \left(\int_{\Gamma_{2\varepsilon}} |w_n|^{2_\alpha^*} \phi_\varepsilon \, dx + \lambda \int_{\Gamma_{2\varepsilon}} |w_n|^{q+1} \phi_\varepsilon \, dx - \kappa_\alpha \int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |\nabla w_n|^2 \phi_\varepsilon \, dxdy \right).$$

Passing to the limit we obtain

$$\lim_{\varepsilon\to 0} \left[\int_{\Gamma_{2\varepsilon}} \phi_\varepsilon \, d\nu + \lambda \int_{\Gamma_{2\varepsilon}} |w_0|^{q+1} \phi_\varepsilon \, dx - \kappa_\alpha \int_{B_{2\varepsilon}^+(x_{k_0})} \phi_\varepsilon \, d\mu \right] = \nu_{k_0} - \kappa_\alpha \mu_{k_0}.$$

Since $\mu_k \geq S(\alpha, N) \nu_k^{\frac{2}{2_\alpha^*}}$, we get that

$$\nu_k = 0 \qquad \text{or} \qquad \nu_k \geq (\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}}, \quad \forall k \in I.$$

Main ideas/steps of the proofs

Lemma 4 If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local (PS)$_c$ condition for any $c < c^*$.

Proof of Lemma 4. Suppose that $\nu_{k_0} \neq 0$ for some $k_0 \in I$. Then

$$
\begin{aligned}
c &= \lim_{n\to\infty} J(w_n) - \frac{1}{2}\langle J'(w_n), w_n\rangle \\
&\geq \frac{\alpha}{2N}\int_\Omega w_0^{2_\alpha^*}dx + \frac{\alpha}{2N}\nu_{k_0} + \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_\Omega w_0^{q+1}dx \\
&\geq \frac{\alpha}{2N}(\kappa_\alpha S(\alpha, N))^{N/\alpha} = c^*,
\end{aligned}
$$

a contradiction. Therefore $\nu_k = 0 \,\forall\, k \in I$, so $u_n \to u_0$ strongly in $L^{2_\alpha^*}(\Omega)$ and we conclude easily. ■

Main ideas/steps of the proofs

Now the **idea** is the following. One consider the minimizers to the Sobolev inequality $u_\varepsilon(x) = \frac{\varepsilon^{(N-\alpha)/2}}{(|x|^2+\varepsilon^2)^{(N-\alpha)/2}}$ and its α-harmonic extension w_ε, not explicit. Consider an appropriate cut-off function ϕ in $\mathcal{C}_\Omega$ centered at the origin (we can assume $0 \in \Omega$), and denote $\psi_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi w_\varepsilon\|}$. Define

$$\Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0,1], X_0^\alpha(\mathcal{C}_\Omega)) : \ \gamma(0) = 0, \ \gamma(1) = t_\varepsilon \psi_\varepsilon\}$$

for some $t_\varepsilon > 0$ such that $\widetilde{J}(t_\varepsilon \psi_\varepsilon) < 0$. And consider the minimax value

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max \widetilde{J}(\gamma(t)) : \ 0 \le t \le 1.$$

Then we are going to prove that for $\varepsilon \ll 1$,

$$c_\varepsilon \le \sup_{t \ge 0} \widetilde{J}(t\psi_\varepsilon) < c^* = \frac{\alpha}{2N} \left(\kappa_\alpha S(\alpha, N)\right)^{N/\alpha}.$$

By the Mountain Pass Theorem, there exists a (PS) sequence $\{w_n\} \subset X_0^\alpha(\mathcal{C}_\Omega)$ verifying $\widetilde{J}(w_n) \to c_\varepsilon < c^*, \quad \widetilde{J}'(w_n) \to 0.$
So by Lemma 4 we finish.

Main ideas/steps of the proofs

Lemma 6 With the above notation, taking $\varepsilon \ll 1$,

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}),$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} c\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ c\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases}$$

and if $r = \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$,

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha.$$

Main ideas/steps of the proofs

Lemma 6 With the above notation, taking $\varepsilon \ll 1$,

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}),$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} c\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ c\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases}$$

and if $r = \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$,

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha.$$

Taking into account that the family u_ε and the Poisson kernel are self-similar $u_\varepsilon(x) = \varepsilon^{\frac{\alpha-N}{2}} u_1(x/\varepsilon)$, $P_y^\alpha(x) = \frac{1}{y^N} P_1^\alpha\left(\frac{x}{y}\right)$, this gives that the family w_ε is also self-similar, more precisely

$$w_\varepsilon(x,y) = \varepsilon^{\frac{\alpha-N}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

We will denote $w_{1,\alpha} = w_1$.

Main ideas/steps of the proofs

Lemma 6 With the above notation, taking $\varepsilon \ll 1$,

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}),$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} c\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ c\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases}$$

and if $r = \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$,

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha.$$

Lemma 7 $w_\varepsilon(x,y) = \varepsilon^{\frac{\alpha-N}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$; $w_{1,\alpha} = w_1$.

$$|\nabla w_{1,\alpha}(x,y)| \leq \frac{c}{y} w_{1,\alpha}(x,y), \quad \alpha > 0,\ (x,y) \in \mathbb{R}^{N+1}_+$$

$$|\nabla w_{1,\alpha}(x,y)| \leq c w_{1,\alpha-1}(x,y), \quad \alpha > 1,\ (x,y) \in \mathbb{R}^{N+1}_+.$$

$$|w_{1,\alpha}(x,y)| \leq C\varepsilon^{N-\alpha}, \quad \frac{1}{2\varepsilon} \leq |(x,y)| \leq \frac{1}{\varepsilon}.$$

Main ideas/steps of the proofs

Lemma 6 With the above notation, taking $\varepsilon \ll 1$,

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}),$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} c\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ c\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases}$$

and if $r = \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$,

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha.$$

Lemma 8 For $\varepsilon \ll 1$,

$$\sup_{t>0} \widetilde{J}(t\psi_\varepsilon) < c^*.$$

Main ideas/steps of the proofs

Lemma 6 With the above notation, taking $\varepsilon \ll 1$,

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}),$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} c\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ c\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases}$$

and if $r = \frac{N+\alpha}{N-\alpha} = 2^*_\alpha - 1$,

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha.$$

Lemma 8 For $\varepsilon \ll 1$,

$$\sup_{t>0} \widetilde{J}(t\psi_\varepsilon) < c^*.$$

For example for $N > 2\alpha$, after some computations,

$$\sup_{t>0} \widetilde{J}(t\psi_\varepsilon) \leq c^* - c\,\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) < c^*. \quad \blacksquare$$