Gradient bounds for elliptic problems singular at the boundary

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Abstract

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and let us denote by $d(x)$ =dist$(x, \partial\Omega)$. We study a class of singular Hamilton-Jacobi equations, arising from stochastic control problems, whose simplest model is

$$-\alpha\Delta u + u + \frac{\nabla u \cdot B(x)}{d(x)} + c(x)|\nabla u|^2 = f(x) \quad \text{in } \Omega,$$

where f belongs to $W^{1,\infty}_{\text{loc}}(\Omega)$ and is (possibly) singular at $\partial\Omega$, $c \in W^{1,\infty}(\Omega)$ (with no sign condition) and the field $B \in (W^{1,\infty}(\Omega))^N$ has the outward direction and satisfies $B \cdot \nu \geq \alpha$ at $\partial\Omega$ (ν is the outward normal). Despite the singularity in the equation, we prove gradient bounds up to the boundary and the existence of a (globally) Lipschitz solution. In some cases, we show that this is the unique bounded solution.

We also discuss the stability of such estimates with respect to α, as α vanishes, obtaining Lipschitz solutions for first order problems with similar features.

The main tool is a refined weighted version of the classical Bernstein's method to get gradient bounds; the key role is played here by the orthogonal transport component of the Hamiltonian.

We also discuss the application of such estimates to a stochastic control problem.