A Not So Long Introduction to the Weak Theory of Parabolic Problems with Singular Data

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by

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CHAPTER 1

Motivations for the problem and basic tools

1.1. Motivations

Here we want to study parabolic equations whose simplest model is the so called *(Nonhomogeneous) Heat Equation*

$$u_t - \Delta u = f,$$

subject to suitable initial and boundary conditions. Here $t > 0$ and $x \in \Omega$ which is an open subset of $\mathbb{R}^N$. The unknown is the function $u : \Omega \times (0, T) \mapsto \mathbb{R}$, where T is a positive, possibly infinity, constant, and Δ is the usual *Laplace Operator* with respect to the space variables, that is

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i},$$

while the function $f : \Omega \times (0, T) \mapsto \mathbb{R}$ is a given datum.

Historically, the study of parabolic equations followed a parallel path with respect to the elliptic theory: so many results of the elliptic framework (harmonic properties, maximum principles, representations of solution, ...) turn out to have a (usually more complicated ♁) parabolic counterpart.

However, unfortunately (or by chance...) the statement *Every elliptic problem becomes parabolic just with time* is, in general, false.

On the other hand, a natural question is the reverse one: *is it true that every parabolic problem turns out to become elliptic with time?* We will try to give an answer to this problem at the end of the last part of the final class (If I can...).

In any case, the physical interpretation is much more *clear* and so these type of equations turned out to admit many many applications in a wide variety of fields as, among others, *Thermodynamics* (ça va sans dire...), *Statistics* (Brownian Motion), *Fluid Mechanics* (Navier-Stokes equations... there is a prize about it[1]), *Finance* (Black-Scholes equation... here there is not a prize anymore... ♁), an so on.

The heat equation can be considered as *diffusion equation* and it was firstly studied to describe the evolution in time of the density u of some quantity such as heat or chemical concentration. If $\omega \subset \Omega$ is a smooth

[1]For instance it is still not known if the N-S equation which describes the flow of air around an airplane has a solution :(

subregion, the rate of change of the total quantity within ω should equal the negative of the flux through $\partial\omega$, that is

$$\frac{d}{dt}\int_\omega u\ dx = -\int_{\partial\omega} F\cdot\nu\ d\sigma,$$

F being the flux density. Thus,

$$u_t = -\mathrm{div}F,$$

as ω is arbitrary. In many applications F turns out to be proportional to ∇u, that leads to $u_t - \lambda\Delta u = 0$, that is the *Heat Equation* for $\lambda = 1$;

$$u_t - \Delta u = 0. \tag{1.1}$$

Let us explicitly remark that the heat equation involves *one* derivative with respect to the time and *two* with respect to x. Consequently, we can easily check that, if u solves (1.1), then so does $u(\lambda x, \lambda^2 t)$, for $\lambda \in \mathbb{R}$; this inhomogeneity suggests that the ratio $\frac{|x|^2}{t}$ is important to study this type of equations and that an explicit radial solution can be searched of the form $u(x,t) = v(\frac{r^2}{t})$, where $r = |x|$, and v is the new unknown.

Let us formally motivate the introduction of the so called *fundamental solution* for (1.1).

It is quicker to search for radial solutions through the invariant scaling $r = |y| = t^{-\frac{1}{2}}|x|$, which yields, after few calculations to derive the radial form of (1.1):

$$\frac{N}{2}v + \frac{1}{2}rv' + v'' + \frac{N-1}{r}v' = 0,$$

which, multiplying the equation by r^{N-1}, turns out to be equivalent to

$$(r^{N-1}v')' + \frac{1}{2}(r^N v)' = 0,$$

that is,

$$r^{N-1}v' + \frac{1}{2}r^N v = a,$$

for some constant a. Now assuming that we look for solutions vanishing at infinity with its derivatives we conclude that $a = 0$. Thus

$$v' = -\frac{1}{2}rv,$$

and so, finally

$$v = be^{-\frac{r^2}{4}}, (b > 0).$$

With a suitable choice of the contants we can write the classic fundamental solution for problem (1.1), that is

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^N, t > 0). \tag{1.2}$$

The fundamental solution can be used to represent solutions for *initial-boundary value problems (Cauchy problems)* of the type

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = g. \end{cases} \tag{1.3}$$

In fact the following result holds true

THEOREM 1.1. *Let $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then*

$$u(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)g(y)\ dy, \tag{1.4}$$

belongs to $C^\infty(\mathbb{R}^N \times (0,\infty))$, solves the equation in (1.3) and

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0), (x_0 \in \mathbb{R}^N).$$

PROOF. [**E**], p. 47. □

If we have a nonhomogeneous smooth forcing term f the representation formula is more complicated (but just a little bit.....) and involves the so called superposition *Duhamel Principle*. In fact, if we assume for simplicity $f \in C_1^2(\mathbb{R}^N \times [0,\infty))$ (i.e. two continuous derivatives in space and one in time) with compact support, then the representation formula for the solution to problem

$$\begin{cases} u_t - \Delta u = f \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = g, \end{cases} \tag{1.5}$$

will be

$$u(x,t) = \int_{\mathbb{R}^N} \Phi(x-y,t)g(y)\ dy + \int_0^t \int_{\mathbb{R}^N} \Phi(x-y,t-s)f(y,s)\ dyds.$$

REMARK 1.2. In view of Theorem 1.1 we sometimes say that the fundamental solution solves

$$\begin{cases} \Phi_t - \Delta \Phi = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ \Phi(x,0) = \delta_0, \end{cases} \tag{1.6}$$

where δ_0 denotes the *Dirac mass* at 0. Notice moreover that, from (1.4), we derive that for nonnegative data $g \neq 0$ the solution turns out to be strictly positive for all $x \in \mathbb{R}^N$, and $t > 0$. This is a *key feature* for parabolic solutions which have the so called *infinite propagation speed*. If the initial temperature is nonnegative and positive somewhere, then at any positive time t the temperature is positive anywhere. This fact turns out to play an essential difference with other types of evolution equations such, for instance, *Hyperbolic Equations*.

As we said many features of harmonic functions are inherited by solutions of the heat equation. Among the others let us just state the *strong maximum principle* which is a consequence of the *parabolic mean value formula* (see [**E**]).

THEOREM 1.3 (Strong Maximum Principle). *Let Ω be a smooth, connected, bounded open set of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. Assume $u \in C_1^2(\Omega \times (0,T]) \cap C(\overline{Q})$ solves the heat equation in Q. Then, if we denote $\Gamma = \overline{Q} \backslash (\Omega \times (0,T])$, we have*

i)
$$\max_{\overline{Q}} u = \max_{\Gamma} u,$$

ii) *If u attains its maximum at $(x_0, t_0) \in Q$, then u is constant in $\overline{\Omega} \times [0, t_0]$.*

PROOF. See [**E**], p. 54. □

Theorem 1.3 has a very suggestive interpretation: with constant data on the boundary, the solution keeps itself constant until *something* happens to change this quiet status (think about a change of boundary conditions from t_0 on). In some sense, a solution behaves in a very intuitive way since *the past turns out to be independent on the future*. This fact is strongly related to the *irreversibility* of the heat equation, that is on the ill-posedness of the *final-boundary value problem*

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,T) \\ u(x,T) = g. \end{cases} \tag{1.7}$$

Looking for solutions to problem (1.7) is, in some sense, equivalent to find out an initial datum such that the corresponding solution of the Cauchy problem coincides with g at time T. However, because of the strong regularization of the solution emphasized by Theorem 1.1, if g is not smooth enough there is no chance to solve (1.7). So this should convince us that the use of the symbol t to denote the last variable of the unknown u is not just a mere chance.

Notation and remarks. Let us spend a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C to denote any positive constant (possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (N, Ω, Q, p, and so on...). In any case such constants never depend on the different indexes having a limit we often introduce, for instance $\varepsilon, \delta, \eta$ that vanish or n, k that go to infinity. Here and in the rest of these notes $\omega(\nu, \eta, \varepsilon, n, h, k)$ will indicate any quantity that vanishes as the parameters go to their (obvious, if not explicitly stressed) limit point with the same order in which they appear, that is, for example

$$\lim_{\nu \to 0} \limsup_{\delta \to 0} \limsup_{n \to +\infty} \limsup_{\varepsilon \to 0} |\omega(\varepsilon, n, \delta, \nu)| = 0.$$

Moreover, for the sake of simplicity, in what follows, the convergences, even if not explicitly stressed, may be understood to be taken possibly up to a suitable subsequence extraction. Finally, for the sake of simplification of the

notation we will indicate the time derivative of a function u with u_t, $\frac{du}{dt}$ or u' depending on the context.

For the convenience of the reader in Appendix A we recall some basic results of *measure and integration theory* we will always assume to be known in the following.

1.2. Functional spaces involving time

Since we want to study an equation involving one derivative in time and two in space, the right functional setting should be C^1 in time and C^2 in space, and, in fact, this is the classical setting we mentioned above.

However, as in the elliptic case, we would like to solve problems with less regular data. Due to this fact, we will deal with the *weak theory* of parabolic problems, so that, to our aims, it would be sufficient a functional setting involving zero derivatives in time (*Lebesgue regularity*) and just one in space (*Sobolev regularity*).

Let us just recall that, if E and F are Banach spaces,then the function

$$\begin{array}{rccc} f: & E & \to & F \\ & x & \mapsto & f(x), \end{array}$$

is said to be *Fréchet differentiable* at $a \in E$, if there exist a linear bounded map D from E to F such that

$$\lim_{\|h\|_E \to 0} \frac{\|f(a+h) - f(a) - D_a(h)\|_F}{\|h\|_E} = 0.$$

Given a real Banach space V, we will denote by $C^\infty(\mathbb{R}; V)$ the space of functions $u : \mathbb{R} \to V$ which are infinitely many times Fréchet differentiable and by $C_0^\infty(\mathbb{R}; V)$ the space of functions in $C^\infty(\mathbb{R}; V)$ having compact support. As we mentioned above, for $a, b \in \mathbb{R}$, $C^\infty([a, b]; V)$ will be the space of the restrictions to $[a, b]$ of functions of $C_0^\infty(\mathbb{R}; V)$, and $C([a, b]; V)$ the space of all continuous functions from $[a, b]$ into V.

We recall that a function $u : [a, b] \to V$ is said to be Lebesgue measurable if there exists a sequence $\{u_n\}$ of step functions (i.e. $u_n = \sum_{j=1}^{k_n} a_j^n \chi_{A_j^n}$ for a finite number k_n of Borel subsets $A_j^n \subset [a, b]$ and with $a_j^n \in V$) converging to u almost everywhere with respect to the Lebesgue measure in $[a, b]$.

Then for $1 \le p < \infty$, $L^p(a, b; V)$ is the space of measurable functions $u : [a, b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p \, dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(a, b; V)$ is the space of measurable functions such that:

$$\|u\|_{L^\infty(a,b;V)} = \sup_{[a,b]} \|u\|_V < \infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [**DL**].

Let us recall that, for $1 \leq p \leq \infty$, $L^p(a,b;V)$ is a Banach space. Moreover, for $1 \leq p < \infty$ and if the dual space V' of V is separable, then the dual space of $L^p(a,b;V)$ can be identified with $L^{p'}(a,b;V')$.

For our purpose V will mainly be either the Lebesgue space $L^p(\Omega)$ or the Sobolev space $W_0^{1,p}(\Omega)$, with $1 \leq p < \infty$ and Ω will be a bounded open set of $\mathbb{R}^N$. Since, in this case, V is separable, we have that $L^p(a,b;L^p(\Omega)) = L^p((a,b)\times\Omega))$, the ordinary Lebesgue space defined in $(a,b)\times\Omega$. Note that $L^p(a,b;W_0^{1,p}(\Omega))$ consists of all functions $u : [a,b]\times\Omega \to \mathbb{R}$ which belong to $L^p((a,b)\times\Omega))$ and such that $\nabla u = (u_{x_1},\dots,u_{x_N})$ belongs to $(L^p((a,b)\times\Omega))^N$. Moreover,

$$\left(\int_a^b \int_\Omega |\nabla u|^p \; dxdt\right)^{\frac{1}{p}}$$

defines an equivalent norm by Poincaré inequality.

Given a function u in $L^p(a,b;V)$ it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a,b;V)$ which is the space of linear continuous functions from $C_0^\infty(a,b)$ into V (see [**Sc**] for further details). In fact, the definition is the following:

$$\langle u_t, \psi\rangle = -\int_a^b u\,\psi_t \; dt\,, \quad \forall \;\; \psi \in C_0^\infty(a,b),$$

where the equality is meant in V. If $u \in C^1(a,b;V)$ this definition clearly coincides with the Fréchet derivative of u. In the following, u_t is said to belong to a space $L^q(a,b;\tilde{V})$ ($\tilde{V}$ being a Banach space) if there exists a function $z \in L^q(a,b;\tilde{V}) \cap \mathcal{D}'(a,b;V)$ such that:

$$\langle u_t, \psi\rangle = -\int_a^b u\,\psi_t \; dt = \langle z, \psi\rangle\,, \quad \forall \;\; \psi \in C_0^\infty(a,b).$$

In the following, we will also use sometimes the notation $\dfrac{\partial u}{\partial t}$ instead of u_t. We recall the following classical embedding result

THEOREM 1.4. *Let H be an Hilbert space such that:*

$$V \underset{dense}{\hookrightarrow} H \hookrightarrow V'\,.$$

x Let $u \in L^p(a,b;V)$ be such that u_t, defined as above in the distributional sense, belongs to $L^{p'}(a,b;V')$. Then u belongs to $C([a,b];H)$.

SKETCH OF THE PROOF. We give a sketch of the proof of this result in the particular case $p = 2$ and $V = H_0^1(\Omega)$ (in this case the *pivot space* H will be $L^2(\Omega)$). A complete proof of Theorem 1.4 can be found in [**DL**]. for simplicity we also choose $a = 0$, and $b = T$.

Extend u to the larger interval $[-\sigma, T+\sigma]$, for $\sigma > 0$, and define the regularizations $u^\varepsilon = \eta^\varepsilon * u$, where η^ε is a mollifier on $\mathbb{R}$. One can easily check that,

$$\begin{cases} u^\varepsilon \to u \ \text{ in } L^2(0,T;H_0^1(\Omega)), \\ u_t^\varepsilon \to u_t \ \text{ in } L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{1.8}$$

Then, for $\varepsilon, \delta > 0$,

$$\frac{d}{dt}\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} = 2\langle u_t^\varepsilon(t) - u_t^\delta(t), u^\varepsilon(t) - u^\delta(t)\rangle_{L^2(\Omega)}.$$

Thus, integrating between s and t we have

$$\begin{aligned} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} &= \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2(\Omega)} \\ &+2\int_s^t \langle u_t^\varepsilon(\tau) - u_t^\delta(\tau), u^\varepsilon(\tau) - u^\delta(\tau)\rangle_{L^2(\Omega)}\, d\tau\,, \end{aligned} \tag{1.9}$$

for all $0 \le s, t \le T$. Now, as a consequence of (1.8), for a.e. $s \in (0,T)$, we have

$$u^\varepsilon(s) \longrightarrow u(s) \quad \text{in } L^2(\Omega).$$

So that, for these s, from (1.9), using both *Cauchy-Schwartz* and *Young's* inequality, we can write

$$\sup_{0\le t\le T} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} \le \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2(\Omega)}$$

$$+\int_0^T \|u_t^\varepsilon(\tau) - u_t^\delta(\tau)\|^2_{H^{-1}(\Omega)} + \|u^\varepsilon(\tau) - u^\delta(\tau)\|^2_{H_0^1(\Omega)}\, d\tau = \omega(\varepsilon,\delta),$$

thanks to (1.8).

So u^ε converges to a function v in $C([0,T];L^2(\Omega))$; since we know that $u^\varepsilon(t) \to u(t)$ for a.e. t, we deduce that $v = u$ a.e.

□

This result also allows us to deduce, for functions u and v enjoying these properties, the integration by parts formula:

$$\int_a^b \langle v, u_t\rangle\, dt + \int_a^b \langle u, v_t\rangle\, dt = (u(b), v(b)) - (u(a), v(a))\,, \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ is the duality between V and V' and $(\cdot,\cdot)$ the scalar product in H. Notice that the terms appearing in (1.10) make sense thanks to Theorem 1.4. The proof of (1.10) relies on the fact that $C_0^\infty(a,b;V)$ is dense in the space of functions $u \in L^p(a,b;V)$ such that $u_t \in L^{p'}(a,b;V')$ endowed with the norm $\|u\| = \|u\|_{L^p(a,b;V)} + \|u_t\|_{L^{p'}(a,b;V')}$, together with the fact that (1.10) is true for u, $v \in C_0^\infty(a,b;V)$ by the theory of integration and derivation in Banach spaces. Note however that in this context (1.10) is subject to the

verification of the hypotheses of Theorem 1.4; if, for instance, $V = W_0^{1,p}(\Omega)$, then

$$W_0^{1,p}(\Omega) \underset{dense}{\hookrightarrow} L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$$

only if $p \geq \frac{2N}{N+2}$; for the sake of simplicity we will often work under this bound, that actually turns out to be only technical to our purposes.

1.2.1. Further useful results. Here we give some further results that will be very useful in what follows; the first one contains a generalization of the integration by parts formula (1.10) where the time derivative of a function is less regular than there; its proof can be found in [**DP**] (see also [**CW**]).

LEMMA 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is compactly supported on $\mathbb{R}$; let us denote $F(s) = \int_0^s f(r)dr$. If $u \in L^p(0,T;W_0^{1,p}(\Omega))$ is such that $u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\overline{Q})$, then we have*

$$\begin{aligned} \int_0^T \langle u_t, f(u)\psi\rangle \, dt &= \int_\Omega F(u(T))\psi(T) \, dx \\ &- \int_\Omega F(u(0))\psi(0) \, dx - \int_Q \psi_t \, F(u) \, dxdt. \end{aligned} \tag{1.11}$$

Now we state three embedding theorems that will play a central role in our work; the first one is the well-known *Gagliardo-Nirenberg embedding theorem* followed by an important consequence of it for the evolution case, while the second one is an Aubin-Simon type result that we state in a form general enough to our purpose; the third one is a useful generalization of Theorem 1.4.

THEOREM 1.6 (Gagliardo-Nirenberg). *Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \geq 1$, $\rho \geq 1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)} \leq C \|\nabla v\|^\theta_{(L^q(\Omega))^N} \|v\|^{1-\theta}_{L^\rho(\Omega)},$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{\rho}.$$

PROOF. See [**N**], Lecture II. □

A consequence of the previous result is the following embedding result:

COROLLARY 1.7. *Let $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, with $q \geq 1$, $\rho \geq 1$. Then $v \in L^\sigma(Q)$ with $\sigma = q\frac{N+\rho}{N}$ and*

$$\int_Q |v|^\sigma \, dxdt \leq C \|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^q \, dxdt. \tag{1.12}$$

PROOF. By virtue of Theorem 1.6, we can write

$$\int_\Omega |v|^\sigma \le C \|\nabla v\|^{\vartheta\sigma}_{L^q(\Omega)} \|v\|^{(1-\vartheta)\sigma}_{L^\rho(\Omega)},$$

that is, integrating between 0 and T

$$\int_0^T \int_\Omega |v|^\sigma \le C \int_0^T \|\nabla v(t)\|^{\vartheta\sigma}_{L^q(\Omega)} \|v(t)\|^{(1-\vartheta)\sigma}_{L^\rho(\Omega)} dt, \tag{1.13}$$

now, since $v \in L^q(0,T;W^{1,q}_0(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, we have

$$\int_0^T \int_\Omega |v|^\sigma \le C \|v\|^{(1-\vartheta)\sigma}_{L^\infty((0,T);L^\rho(\Omega))} \int_0^T \|\nabla v(t)\|^{\sigma\vartheta}_{L^q(\Omega)} dt.$$

Now we choose

$$\vartheta = \frac{q}{\sigma} = \frac{N}{N+\rho}$$

so that

$$\sigma\vartheta = q, \quad (1-\vartheta)\sigma = \frac{q\rho}{N},$$

and (1.13) becomes

$$\int_0^T \int_\Omega |v|^\sigma \le C \|v\|^{\frac{q\rho}{N}}_{L^\infty((0,T);L^\rho(\Omega))} \int_0^T \|\nabla v(t)\|^q_{L^q(\Omega)} dt,$$

that is

$$\int_Q |v|^\sigma \le C \|v\|^{\frac{q\rho}{N}}_{L^\infty((0,T);L^\rho(\Omega))} \int_Q |\nabla v|^q.$$

□

REMARK 1.8. Let us explicitly remark that Corollary 1.7 gives us a little gain on the *a priori* summability of the involved function (actually this is not a consequence of Gagliardo-Nirenberg inequality, it is a consequence of a *Petitta's inequality*). As an example, let us think about a function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$; in this case the solution turns out to belong to $L^{2+\frac{4}{N}}(Q)$.

THEOREM 1.9. *Let u^n be a sequence bounded in $L^q(0,T;W^{1,q}_0(\Omega))$ such that u^n_t is bounded in $L^1(Q) + L^{s'}(0,T;W^{-1,s'}(\Omega))$ with $q,s>1$, then u^n is relatively strongly compact in $L^1(Q)$, that is, up to subsequences, u^n strongly converges in $L^1(Q)$ to some function u.*

PROOF. See [**Si**], Corollary 4. □

Let us define, for every $p>1$, the space S^p as

$$S^p = \{u \in L^p(0,T;W^{1,p}_0(\Omega)); u_t \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))\}, \tag{1.14}$$

endowed with its natural norm

$$\|u\|_{S^p} = \|u\|_{L^p(0,T;W^{1,p}_0(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}.$$

We have the following trace result:

THEOREM 1.10. *Let $p > 1$, then we have the following continuous injection*

$$S^p \hookrightarrow C(0, T; L^1(\Omega)).$$

PROOF. See [**Po**], Theorem 1.1. □

CHAPTER 2

Weak solutions

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, $t > 0$; we denote by Q_t the cylinder $\Omega \times (0,t)$. If $t = T$ we will often write Q for Q_T. In this chapter we are interested in the study existence, uniqueness, and regularity of the solution of the linear parabolic problem

$$\begin{cases} u_t + L(u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.1}$$

where

$$L(u) = -\mathrm{div}(A(x,t)\nabla u),$$

and A is a matrix with bounded, measurable entries, such that

$$|A(x,t)\xi| \leq \beta|\xi|, \tag{2.2}$$

for any $\xi \in \mathbb{R}^N$, with $\beta > 0$, and

$$A(x,t)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{2.3}$$

for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$. As we will see such results strongly depend on the regularity of the data f, u_0 and A.

We first deal existence, uniqueness and (weak) regularity for linear problems in the framework of Hilbert spaces, that is $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in H_0^1(\Omega)$. For such data the solution is supposed to be in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, with $u_t \in L^2(0,T;H^{-1}(\Omega))$. Moreover, we expect the solutions to belong to $C(0,T;L^2(\Omega))$ to give sense at the initial value u_0.

Indeed if we formally multiply the equation in (2.1) by u and using (2.3), integrating on Ω and between 0 and t (here $0 < t \leq T$), we obtain, using also Young's inequality,

$$\begin{aligned} \int_0^t \langle u_t, u\rangle + \alpha \int_Q |\nabla u|^2 &\leq \int_0^t \|f\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)} \\ &\leq \|f\|_{L^2(0,T;H^{-1}(\Omega))} \|u\|_{L^2(0,T;H_0^1(\Omega))} \\ &\leq \frac{1}{2\alpha}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;H_0^1(\Omega))}. \end{aligned}$$

Which, thanks to the fact that

$$\int_0^t \langle u_t, u\rangle = \int_0^t \frac{1}{2}\frac{d}{dt}u^2,$$

using (1.10) yields

$$\frac{1}{2}\int_\Omega u^2(t) + \frac{\alpha}{2}\int_{Q_t} |\nabla u|^2 \leq C(\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u_0\|^2_{L^2(\Omega)}).$$

Now, since the right hand side does not depend on t we easily deduce that the same inequality holds true for any $0 \leq t \leq T$, and so

$$\begin{aligned} &\frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;H^1_0(\Omega))} \\ &\leq C(\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|u_0\|^2_{L^2(\Omega)}), \end{aligned} \tag{2.4}$$

which thanks to Theorem 1.4 it gives the desired regularity result.

REMARK 2.1. Let us stress the fact that, as a difference with the elliptic case, here it is not so easy to face the problem with a *Lax-Milgram* type approach because of the features of the involved functional spaces and of the operator itself. Indeed, roughly speaking, the supposed involved *bilinear form* would turn out to be, for instance, not continuous on $L^2(0,T;H^1_0(\Omega))$, not coercive on $W = \{u \in L^2(0,T;H^1_0(\Omega)), u_t \in L^2(0,T;H^{-1}(\Omega))\}$, and $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ is not an Hilbert space.

2.1. Galerkin Method: Existence and uniqueness of a weak solution

Let us first give our definition for weak solutions to problem (2.1)

DEFINITION 2.2. We say that a function $u \in L^2(0,T;H^1_0(\Omega))$, such that $u_t \in L^2(0,T;H^{-1}(\Omega))$ is a *weak solution* for problem (2.1) if

$$\int_0^T \langle u', \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_Q A(x,t)\nabla u\cdot\nabla\varphi = \langle f, \varphi\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H^1_0(\Omega))}, \tag{2.5}$$

for all $\varphi \in L^2(0,T;H^1_0(\Omega))$ such that $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$, $\varphi(T) = 0$, and $u(x,0) = u_0$ in the sense of $L^2(\Omega)$.

REMARK 2.3. Observe that, taking into account Theorem 1.4, we can easily see that all terms in Definition 2.2 turn out to make sense.

Moreover, as shown in [**E**] (actually it is not so difficult to check), if $f \in L^2(Q)$, u is a weak solution for problem (2.1) if and only if

$$\langle u'(t), v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_\Omega A(x,t)\nabla u(t)\nabla v = \int_\Omega f(t)v, \tag{2.6}$$

for any $v \in H^1_0(\Omega)$ and a.e. in $0 \leq t \leq T$, with $u(0) = u_0$. We will often denote by $\langle\cdot,\cdot\rangle$ the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, while $(\cdot,\cdot)$ will be occasionally used to indicate the inner product in $L^2(\Omega)$.

Now we state our first existence and uniqueness result. Here, for the sake of simplicity, we choose $f \in L^2(Q)$. We will use the so called *Galerkin Method* which relies on the approximation of our problem by mean of finite dimensional problems.

THEOREM 2.4. *Let $f \in L^2(Q)$, and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution for problem (2.1).*

PROOF. We will consider a sequence of functions $w_k(x)$, $(k = 1, \ldots)$ which satisfy

i) $\{w_k\}$ is an *orthogonal* basis of $H_0^1(\Omega)$,
ii) $\{w_k\}$ is an *orthonormal* basis of $L^2(\Omega)$.

Observe that the construction of such a sequence is always possible; as an example (see [**E**]) we can take $w_k(x)$ as a sequence for $-\Delta$ in $H_0^1(\Omega)$ (after a suitable normalization).

Fix now an integer m. We will look for a function $u_m : [0, T] \mapsto H_0^1(\Omega)$ of the form

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k, \tag{2.7}$$

and we want to select the coefficients such that

$$d_m^k(0) = (u_0, w_k) \quad (k = 1, \ldots, m) \tag{2.8}$$

and

$$(u_m', w_k) + \int_\Omega A(x,t) \nabla u_m \cdot \nabla w_k = (f, w_k), \tag{2.9}$$

a.e. on $0 \le t \le T$, $k = 1, \ldots, m$. In other words, we look for the solutions of the *projections* of problem (2.1) to the finite dimensional subspaces of $H_0^1(\Omega)$ spanned by $\{w_k\}(k = 1, \ldots, m)$.

During this proof, for the convenience of the reader we will use the following notation

$$a(\psi, \varphi, t) \equiv \int_\Omega A(x,t) \nabla \psi \cdot \nabla \varphi.$$

We split the proof of this result in four steps.

Step 1. Construction of approximate solutions.

Assume that u_m has the structure (2.7); since w_k is orthonormal in $L^2(\Omega)$, then

$$(u_m', w_k) = \frac{d}{dt} d_m^k(t),$$

and

$$a(u_m, w_k, t) = \sum_{l=1}^{m} e^{kl}(t) d_m^l(t),$$

where $e^{kl}(t) = a(w_l, w_k, t)(k, l = 1, \dots, m)$. Finally, if $f^k = (f(t), w_k)$ is the projection of the datum f, then (2.9) becomes the linear system of ODE

$$\frac{d}{dt} d_m^k(t) + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t),$$

for $k = 1, \dots, m$, subject to the initial condition (2.8). According to the standard existence theory for ordinary differential equations, there exists a unique absolutely continuous function $d_m(t) = (d_m^l, \dots, d_m^l)$ satisfying (2.8) and the ODE for a.e. $0 \le t \le T$. Then u_m is the desired approximate solution since it turns out to solve (2.9).

Step 2. Energy Estimates.

Multiply the equation (2.9) by $d_m^k(t)$ and sum over k between 1 and m. Then, integrating between 0 and T, we find, recalling that $(u_m', u_m) = \frac{1}{2}\frac{d}{dt} u_m^2$, and using (2.3)

$$\frac{1}{2} \int_0^T \frac{d}{dt} \int_\Omega u_m^2 + \alpha \int_Q |\nabla u_m|^2 \le \int_Q f u_m.$$

Thus, reasoning as in the proof of (2.4) we can check that

$$\|u_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_m\|^2_{L^2(0,T;H_0^1(\Omega))} \le C(\|f\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(\Omega)}), \tag{2.10}$$

(here we also used that $\|u_m(0)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$).

Finally, using the fact that w_k is an orthogonal basis in $H_0^1(\Omega)$, we can fix any $v \in H_0^1(\Omega)$ such that $\|v\|_{H_0^1(\Omega)} \le 1$, and deduce from the equation (2.9), after a few easy calculations

$$|\langle u_m', v\rangle| \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H_0^1(\Omega)}),$$

that is

$$\|u_m'\|_{H^{-1}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H_0^1(\Omega)}),$$

whose square integrated between 0 and T, gathered together with (2.10), yields

$$\|u_m'\|^2_{L^2(0,T;H^{-1}(\Omega))} \le C(\|f\|^2_{L^2(Q)} + \|u_0\|^2_{L^2(\Omega)}). \tag{2.11}$$

Step 3. Existence of a solution.

From (2.10) we deduce that there exists a function $u \in L^2(0,T;H_0^1(\Omega))$, such that u_m converges weakly to u in $L^2(0,T;H_0^1(\Omega))$; moreover u_m' weakly converges to some function η in $L^2(0,T;H^{-1}(\Omega))$ (one can easily check by using its definition that $\eta = u'$). Then, we can pass to the limit in the weak formulation of u_m, that is in

$$\int_0^T \langle u_m', \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_Q A(x,t)\nabla u_m \cdot \nabla\varphi = \langle f, \varphi\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))}. \tag{2.12}$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega))$, such that $\varphi' \in L^2(0,T;H^{-1}(\Omega))$, with $\varphi(T) = 0$, to obtain (2.5).

To check that the initial value is achieved we use (1.10) in (2.5) and (2.12), obtaining respectively

$$-\int_0^T \langle \varphi', u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u(0)\varphi(0) + \int_Q A(x,t)\nabla u\cdot\nabla\varphi = \int_Q f\varphi,$$

and

$$-\int_0^T \langle \varphi', u_m\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u_m(0)\varphi(0) + \int_Q A(x,t)\nabla u_m\cdot\nabla\varphi = \int_Q f\varphi.$$

Now, since $u_m(0)\to u_0$ in $L^2(\Omega)$, and $\varphi(0)$ is arbitrary we conclude that $u_0 = u(0)$.

Step 4. Uniqueness of the solution.

Let u and v be two solutions of problem (2.1) in the sense of Definition 2.2; if we take $u-v$ as test function in the weak formulation for both u and v (by a density argument we can see that this function can be chosen as test in (2.5) even if it does not satisfy, a priori, $(u-v)(T)=0$). By subtracting, using that $(u-v)(0)=0$, we obtain

$$\frac{1}{2}\int_\Omega |u-v|^2(T) + \int_Q |\nabla(u-v)|^2 \le 0,$$

that implies $u=v$ a.e. in Q.

□

CHAPTER 3

Regularity

3.1. Regularity for finite energy solutions and regularity $L^r(L^q)$

In this section, we will be concerned with regularity and existence results for solutions of parabolic problem (2.1).

Let us state a first improvement on the regularity of such solutions.

THEOREM 3.1 (Improved regularity). *Let $u_0 \in H_0^1(\Omega)$ and $f \in L^2(Q)$. Then the weak solution u of (2.1) satisfies*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \quad u' \in L^2(Q),$$

with continuous estimates with respect to the data.

PROOF. [**E**], Theorem 5, pag. 360. □

The previous statement gives a first standard regularity result in the Hilbertian case, but what happens if we know something more (or something less) on the datum f?; for instance, what is the *best Lebesgue space* which the solution turns out to belong to?

We will prove the following

THEOREM 3.2. *Assume (2.2), (2.3), $u_0 \in L^2(\Omega)$, and let f belong to$L^r(0,T;L^q(\Omega))$ with r and q belonging to $[1,+\infty]$ and such that*

$$\frac{1}{r} + \frac{N}{2q} < 1\,. \tag{3.13}$$

Then there exists a weak solution of (2.1) belonging to $L^\infty(Q)$. Moreover there exists a positive constant d, depending only from the data (and hence independent on u), such that

$$\|u\|_{L^\infty(Q)} \le d. \tag{3.14}$$

Notice that assumption (3.13) implies that $r \in (1,+\infty]$ and $q \in \left(\frac{N}{2},+\infty\right]$.

To give an idea, let us represent the summability of the datum $f \in L^r(0,T;L^q(\Omega))$ in a diagram with axes $\dfrac{1}{q}$ and $\dfrac{1}{r}$. Since r, $q \in [1,+\infty]$, then all the possible cases of summability are inside of the square $[0,1]\times[0,1]$ (we use the notation $\dfrac{1}{\infty} = 0$).

If f belongs to $L^r(0,T;L^q(\Omega))$ where r and q are large enough, that is, if

$$\frac{1}{r}+\frac{N}{2q}<1 \quad \text{(zone 1 in Figure 1 below)}, \tag{3.15}$$

then every weak solution u belonging to

$$V_2(Q)\equiv L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$$

belongs also to $L^\infty(Q)$ (see Theorem 3.2 above). This fact was proved by Aronson and Serrin in the nonlinear case (see [**AS**]), while can be found in the linear setting in some earlier papers as, among the others, [**LU**] and [**A**].

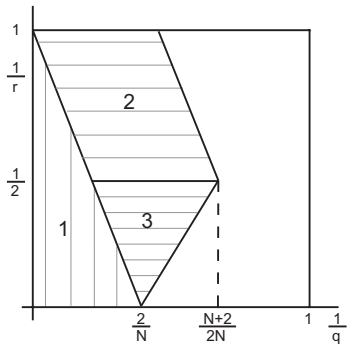


FIGURE 1. Classical regularity results.

On the other hand, if r and q do not satisfy (3.15) but satisfy

$$2<\frac{2}{r}+\frac{N}{q}\le \min\left\{2+\frac{N}{r},2+\frac{N}{2}\right\}, \qquad r\ge 1\,, \tag{3.16}$$

that is, zone 2 and 3 in the figure above, then Ladyženskaja, Solonnikov and Ural'ceva (see Theorem 9.1, cap. 3 in [**LSU**]) proved that any weak solution of (2.1) belonging to $V_2(Q)$ satisfies also

$$|u(x,t)|^\gamma\in V_2(Q), \tag{3.17}$$

where γ is a constant greater than one that is given by an explicit formula in terms of N, r and q. Notice that the regularity (3.17) and the Theorem 1.7 imply that

$$u\in L^s(Q), \qquad s=\frac{(N+2)qr}{Nr+2q-2qr}\,. \tag{3.18}$$

A natural question that arises is whether there exists at least a solution of (2.1) belonging to $V_2(Q)$ if the summability exponents (r,q) of f verify (3.16).

If (3.16) holds with $r\ge 2$ (zone 3 in Figure 1) then the function f belongs also to the space $L^2(0,T;H^{-1}(\Omega))$, so that it is very easy to deduce the existence of at least such a weak solution.

If otherwise (3.16) holds with $r < 2$ (zone 2 in Figure 1) then f doesn't belong to $L^2(0,T;L^{(2^*)'}(\Omega))$, but again there exists at least a solution of (2.1) belonging to $V_2(Q)$ as proved in [**LSU**] for linear operators (see [**BDGO**] for more general nonlinear operators).

Indeed in [**LSU**] (Theorem 4.1 cap. 3) it is proved the previous existence result when the summability exponent of f verifies

$$\frac{1}{r}+\frac{N}{2q}=1+\frac{N}{4} \quad q\in[\frac{2N}{N+2},2], \quad r\in[1,2],$$

but this implies that the result is true for every choice of exponents (r,q) verifying

(3.19)

$$\frac{1}{r}+\frac{N}{2q}\le 1+\frac{N}{4}, \quad q\ge\frac{2N}{N+2} \quad \text{(see zone 2, 3 and 4 in figure 2 below).}$$

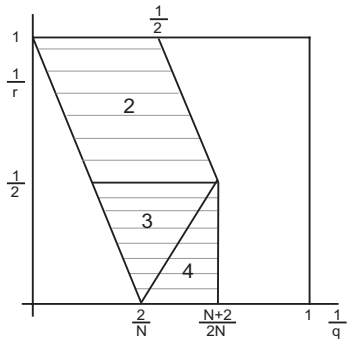


FIGURE 2. Existence results, zone 2, 3 and 4.

Notice that the previous zone includes strictly zone 2 and 3 of Figure 1. What happen in the remaining zone (i.e zone 4 in figure 2) where, as just said, there exists at least a weak solution belonging to $V_2(Q)$?
Moreover, what happens outside of these zones? Are there other zones where there exist $V_2(Q)$ solutions?
In addition, where it is not reasonable to expect solutions in $V_2(Q)$, as for example when r and q are not too big, (that is just for $q<(2^*)'$), and also when this regularity occurs, which is the starting regularity which ensures more summability properties of the solutions (of all the solutions) and which is the possibly optimal Lebesgue summability exponent of the solutions?

Recall that outside the zone 1 in figure 1 it is possible to show examples of unbounded solutions: does the same happen with the previous regularity results?

Surprisingly there aren't exhaustive answers to these questions in literature. We just mention a regularity result concerning data f belonging to $L^2(0,T;H^{-1}(\Omega))\cap L^r(0,T;L^q(\Omega))$ and solutions in the energy space

$L^2(0,T;H_0^1(\Omega))$ (see [**GM**]). The remaning open questions has been recently faced in [**BPP**].

In order to prove that the solutions of (2.1) are bounded when the summability exponents of f are in the zone 1 in the figure 1, we enunciate a very well known lemma due to Guido Stampacchia ([**S**]).

LEMMA 3.3. *Let us suppose that φ is a real, non negative and non increasing function verifying*

$$\varphi(h) \leq \frac{C}{(h-k)^\delta}\ [\varphi(k)]^\nu \quad \forall\ h > k > k_0, \tag{3.20}$$

where C and δ are positive constants and $\nu > 1$. Then there exists a positive constant d such that

$$\varphi(k_0 + d) = 0.$$

Let us give the proof of Theorem 3.2

PROOF OF THEOREM 3.2. Let us use the classical notation

$$A_k^t = \{x \in \Omega :\ |u_n(x,t)| > k\}, \tag{3.21}$$

and, for simplicity let us prove the result for $u_0 = 0$ and positive solutions (it suffices to suppose $f \geq 0$). We can write

$$\frac{1}{r} + \frac{N}{2q} = 1 - \chi_1,\ \chi_1 \in (0,1). \tag{3.22}$$

Taking $G_k(u)$ as a test function in (2.1), integrating in $(0,t_1] \times \Omega$, where $t_1 \leq T$ will be chosen later and using assumption (2.3) we get

$$\frac{1}{2}\int_\Omega G_k(u)^2(t_1)dx + \alpha\int_0^{t_1}\int_\Omega |\nabla G_k(u)dxdt \leq$$
$$\int_0^{t_1}\int_{A_k^t} fG_k(u).dxdt.$$

Therefore we get

$$C_0\left[\|G_k(u)\|^2_{L^\infty(0,t_1;L^2(\Omega))} + \|\nabla G_k(u)\|^2_{L^2(0,t_1;L^2(\Omega)}\right]$$
$$\leq \int_0^{t_1}\int_{A_k^t} fG_k(u)\,dx\,dt, \tag{3.23}$$

Using that $s \leq s^2 + 1$, we get that the right hand of (3.23) can be estimated as follows,

$$\int_0^{t_1}\int_{A_k^t} fG_k(u)\,dx\,dt \leq \tag{3.24}$$

$$\int_0^{t_1}\int_{A_k^t} fG_k(u)^2\,dx\,dt + \int_0^{t_1}\int_{A_k^t} f\,dx\,dt. \tag{3.25}$$

Let us estimate the two integrals in the righ-hand side of the previous inequality (3.24).
Applying Hölder's inequality we obtain

$$\int_0^{t_1}\int_{A_k^t} fG_k(u)^2\,dx\,dt \le C_f\left[\int_0^{t_1}\left(\int_{A_k^t} G_k(u)^{2q'}\,dx\right)^{\frac{r'}{q'}}dt\right]^{\frac{1}{r'}}$$
$$\le C_f\left(\int_0^{t_1}\|G_k(u)\|^{2r'}_{L^{2q'}}\,dt\right)^{\frac{1}{r'}},$$

where $C_f = \|f\|_{L^r(0,T;L^q(\Omega))}$. Let us define

$$\bar r = 2r',\quad \bar q = 2q',\quad \hat r = \bar r(1+\chi),\quad \hat q = \bar q(1+\chi),\quad \chi = \frac{2\chi_1}{N}, \tag{3.26}$$

such that $\dfrac{1}{\hat r} + \dfrac{N}{2\hat q} = \dfrac{N}{4}$, with χ_1 as in (3.22), and denote

$$\mu(k) = \int_0^{t_1}\left(\int_{A_k^t} dx\right)^{\frac{\hat r}{\hat q}} dt. \tag{3.27}$$

Thus, applying Hölder's inequality we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_{A_k^t} fG_k(u)^2\,dx\,dt \le C_f\left(\int_0^{t_1}\|G_k(u)\|^{\bar r}_{L^{\bar q}(A_k^t)}\,dt\right)^{\frac{2}{\bar r}}\\
&= C_f\left[\int_0^{t_1}\left(\int_{A_k^t} G_k(u)^{\frac{\hat q}{1+\chi}}\,dx\right)^{\frac{\hat r}{\hat q}}dt\right]^{\frac{2(1+\chi)}{\hat r}}\\
&\le C_f\left[\int_0^{t_1}\left(\int_{A_k^t} G_k(u)^{\hat q}\,dx\right)^{\frac{\hat r}{\hat q(1+\chi)}}\left(\int_{A_k^t} dx\right)^{\frac{\chi}{1+\chi}\frac{\hat r}{\hat q}}dt\right]^{\frac{2(1+\chi)}{\hat r}}\\
&\le C_f\left[\int_0^{t_1}\left(\int_{A_k^t} G_k(u)^{\hat q}\,dx\right)^{\frac{\hat r}{\hat q}}dt\right]^{\frac{2}{\hat r}}\left[\int_0^{t_1}\left(\int_{A_k^t} dx\right)^{\frac{\hat r}{\hat q}}dt\right]^{\frac{2\chi}{\hat r}}\\
&= C_f\|G_k(u)\|^2_{L^{\hat r}(0,t_1;L^{\hat q}(\Omega))}\cdot\mu(k)^{\frac{2\chi}{\hat r}}.
\end{aligned} \tag{3.28}$$

We estimate now the last term in (3.28). Thanks to interpolation's inequality of Theorem 1.6 applied with $\eta = \hat q$, $\rho = h = 2$ and thus $\theta = \left(\dfrac{N}{2} - \dfrac{N}{\hat q}\right)$, we

get

$$\|G_k(u)\|^2_{L^{\hat{r}}(0,t_1;L^{\hat{q}}(\Omega))} \leq$$
$$C_1 \left(\int_0^{t_1} \|\nabla G_k(u)\|^{\hat{r}\theta}_{L^2(\Omega)} \cdot \|G_k(u)\|^{(1-\theta)\hat{r}}_{L^2(\Omega)}\, dt \right)^{\frac{2}{\hat{r}}}$$
$$\leq C_1 \|G_k(u)\|^{2(1-\theta)}_{L^\infty(0,t_1;L^2(\Omega))} \cdot \left(\int_0^{t_1} \|\nabla G_k(u)\|^{\hat{r}\theta}_{L^2(\Omega)} dt \right)^{\frac{2}{\hat{r}}}.$$

Thus applying Young's inequality we get

$$\|G_k(u)\|^2_{L^{\hat{r}}(0,t_1;L^{\hat{q}}(\Omega))} \leq$$
$$C_1(1-\theta)\|G_k(u)\|^2_{L^\infty(0,t_1;L^2(\Omega))} + C_1\theta \left(\int_0^{t_1} \|\nabla G_k(u)\|^{\hat{r}\theta}_{L^2(\Omega)} dt \right)^{\frac{2}{\hat{r}}\frac{1}{\theta}}.$$

By assumption (3.22) and using (3.26) it follows that $\hat{r}\theta = 2$ and thus we obtain

$$\|G_k(u)\|^2_{L^{\hat{r}}(0,t_1;L^{\hat{q}}(\Omega))} \leq C_2 \|G_k(u)\|^2_{V((0,t_1)\times\Omega)}, \tag{3.29}$$

where we have set

$$\|G_k(u)\|^2_{V((0,t_1)\times\Omega)} = \|G_k(u)\|^2_{L^\infty(0,t_1;L^2(\Omega))} + \|\nabla G_k(u)\|^2_{L^2(0,t_1;L^2(\Omega))},$$

and $C_2 = \min\{C_1(1-\theta), C_1\theta\}$. Applying (3.29) in (3.28) we obtain

$$\int_0^{t_1} \int_{A_k^t} f G_k(u)^2 \, dx \, dt \leq C_f C_2 \mu(k)^{\frac{2\chi}{\hat{r}}} \|G_k(u)\|^2_{(V(0,t_1)\times\Omega)} \tag{3.30}$$

and thus we have estimated the first integral in the right-hand side of (3.24). On the other hand, the second term on the right hand in (3.24) satisfies

$$\int_0^{t_1} \int_{A_k^t} f \, dx \, dt$$
$$\leq \left[\int_0^{t_1} \left(\int_{A_k^t} |f|^q \, dx \right)^{\frac{r}{q}} dt \right]^{\frac{1}{r}} \cdot \left[\int_0^{t_1} \left(\int_{A_k^t} dx \right)^{\frac{r'}{q'}} dt \right]^{\frac{1}{r'}} \tag{3.31}$$
$$= C_f \mu(k)^{\frac{2(1+\chi)}{\hat{r}}},$$

where C_f is as before. Putting together both (3.28) and (3.31) in (3.23) we conclude that

$$C_0 \|G_k(u)\|^2_{V((0,t_1)\times\Omega)} \leq C_f C_2 \|G_k(u)\|^2_{V((0,t_1)\times\Omega)} \cdot \mu(k)^{\frac{2\chi}{\hat{r}}} + C_f \mu(k)^{\frac{2(1+\chi)}{\hat{r}}}.$$

Let us choose t_1 small enough, i.e., such that

$$C_f C_2 t_1^{\frac{2\chi}{\hat{r}}} |\Omega|^{\frac{2\chi}{\hat{q}}} < C_0. \tag{3.32}$$

Using again (3.29) we obtain

$$C_3C_2^{-1}\|G_k(u)\|^2_{L^{\hat r}(0,t_1;L^{\hat q}(\Omega))} \le C_3\|G_k(u)\|^2_{V((0,t_1)\times\Omega)} \le C_f\mu(k)^{\frac{2(1+\chi)}{\hat r}}, \tag{3.33}$$

where $C_3 = C_0 - C_fC_2t_1^{\frac{2\chi}{\hat r}}|\Omega|^{\frac{2\chi}{\hat q}}$. If $h > k > 0$, we then deduce

$$|||G_k(u_n)|||^2_{L^{\hat r}(0,t_1;L^{\hat q}(\Omega))} \ge (h-k)^2\mu(h)^{\frac{2}{\hat r}}$$

that with (3.33) gives

$$\mu(h) \le \frac{C_4}{(h-k)^{\hat r}}\mu(k)^{1+\chi},$$

where $C_4 = (C_fC_2C_3)^{\frac{\hat r}{2}}$. Applying Lemma 3.3, we conclude that there exists a constant d, depending only on q, r, $\|f\|_{L^r(0,T;L^q(\Omega)}$, γ and α, such that $\mu(d) = 0$, that is

$$\|u\|_{L^\infty(([0,t_1]\times\Omega))} \le d.$$

Iterating this procedure in the sets $\Omega\times[t_1,2t_1],\cdots,\Omega\times[jt_1,T]$, where $T - jt_1 \le t_1$, (notice that the process works since in all these sets (3.32) is verified), we can conclude that (3.14) holds true. □

CHAPTER 4

Distributional solutions

In the previous Chapter we proved existence and regularity results for problem (2.1) with (*in some sense*) regular data. What happens if the hypotheses on the data are, even drastically, relaxed? Let us, for instance, face the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t)\nabla u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.34}$$

with $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$, and $A(x,t)$ satisfying (2.2) and (2.3).

Problem (4.34) turns out to admit a distributional solution, that is a function $u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap C(0,T;L^1(\Omega))$ such that

$$-\int_Q \phi_t \; u + \int_Q A(x,t)\nabla u \cdot \nabla \phi = \int_Q f\phi,$$

for any $\phi \in \mathcal{D}(Q)$, and $u(0) = u_0$ in the sense of $L^1(\Omega)$.

That is, we want to prove the following

THEOREM 4.1. *Let $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$, and $A(x,t)$ satisfying (2.2) and (2.3). Then, problem (4.34) has a distributional solution.*

PROOF. We first approximate the data with smooth functions $u_{0,n}$ and f_n which converge, respectively, to u_0 in $L^1(\Omega)$ and to f in $L^1(Q)$. Moreover we can choose such functions such that

$$\|f_n\|_{L^1(Q)} \le \|f\|_{L^1(Q)}, \qquad \|u_{0,n}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}.$$

We want to look for some *a priori estimates* concerning the sequence u^n of weak solutions of the approximate problems

$$\begin{cases} u_t^n - \operatorname{div}(A(x,t)\nabla u^n) = f_n & \text{in } \Omega \times (0,T), \\ u^n(0) = u_{0,n} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.35}$$

Observe that a unique weak solution exists for problem (4.35) thanks to Theorem 2.4.

Let us fix n and let us take $T_k(u^n)$ in the weak formulation for u^n (this can be made rigorous thanks to an easy approximation argument, see also

Remark 2.3); here $T_k(s) = \max(-k, \min(s,k))$. Integrating between 0 and t, we get

$$\begin{aligned} &\int_\Omega \Theta_k(u^n)(t) + \alpha \int_Q |\nabla T_k(u^n)|^2 \, dxdt \\ &\leq k(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) = Ck, \end{aligned} \tag{4.36}$$

where Θ_k denotes the primitive function of T_k.

Therefore, for every fixed $k > 0$, from the first term on the left hand side of (4.36), since $\Theta_k(s) \geq ks - 1$ (for $s \geq 0$), we get that u^n is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, while from the second one we have that $T_k(u^n)$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))$.

We can improve this kind of estimate by using the *Gagliardo-Nirenberg* inequality (see Corollary 1.7). Indeed in this way we get

$$\int_Q |T_k(u^n)|^{2+\frac{2}{N}} \, dxdt \leq Ck \tag{4.37}$$

and so, we can write

$$\begin{aligned} k^{2+\frac{2}{N}} \mathrm{meas}\{|u^n| \geq k\} &\leq \int_{\{|u^n|\geq k\}} |T_k(u^n)|^{2+\frac{2}{N}} \, dxdt \\ &\leq \int_Q |T_k(u^n)|^{2+\frac{2}{N}} \, dxdt \leq Ck; \end{aligned}$$

then,

$$\mathrm{meas}\{|u^n| \geq k\} \leq \frac{C}{k^{1+\frac{2}{N}}}\,. \tag{4.38}$$

Therefore, the sequence u^n is uniformly bounded in the *Marcinkiewicz* space $M^{1+\frac{2}{N}}(Q)$; that implies that u^n is uniformly bounded in $L^m(Q)$ for all $1 \leq m < 1+\frac{2}{N}$ (for further properties of Marcinkiewicz spaces see Appendix A).

We are interested about a similar estimate on the gradients of functions u^n; let us emphasize that this estimate holds true for all functions satisfying (4.36), so, for the convenience of the reader, we will omit the index n. First of all, observe that

$$\mathrm{meas}\{|\nabla u| \geq \lambda\} \leq \mathrm{meas}\{|\nabla u| \geq \lambda; |u| \leq k\} + \mathrm{meas}\{|\nabla u| \geq \lambda; |u| > k\}. \tag{4.39}$$

With regard to the first term to the right hand side of (4.39) we have

$$\begin{aligned} \mathrm{meas}\{|\nabla u| \geq \lambda; |u| \leq k\} &\leq \frac{1}{\lambda^2} \int_{\{|\nabla u|\geq\lambda; |u|\leq k\}} |\nabla u|^2 \, dx \\ &\leq \frac{1}{\lambda^2} \int_{\{|u|\leq k\}} |\nabla u|^2 \, dx = \frac{1}{\lambda^2} \int_Q |\nabla T_k(u)|^2 \, dx \leq \frac{Ck}{\lambda^2}; \end{aligned} \tag{4.40}$$

while for the last term in (4.39), thanks to (4.38), we can write

$$\text{meas}\{|\nabla u| \geq \lambda; |u| > k\} \leq \text{meas}\{|u| \geq k\} \leq \frac{\overline{C}}{k^\sigma},$$

with $\sigma = 1 + \frac{2}{N}$. So, finally, we get

$$\text{meas}\{|\nabla u| \geq \lambda\} \leq \frac{\overline{C}}{k^\sigma} + \frac{Ck}{\lambda^2},$$

and we can have a better estimate by taking the minimum over k of the right hand side; the minimum is achieved for the value

$$k_0 = \left(\frac{\sigma C}{\overline{C}}\right)^{\frac{1}{\sigma+1}} \lambda^{\frac{2}{\sigma+1}},$$

and so we get the desired estimate

$$\text{meas}\{|\nabla u| \geq \lambda\} \leq C\lambda^{-\gamma} \tag{4.41}$$

with $\gamma = 2(\frac{\sigma}{\sigma+1}) = \frac{N+2}{N+1}$.

Then, coming back to our case, we have found that, for every $n \geq 1$, $|\nabla u^n|$ is equi-bounded in $M^\gamma(Q)$, with $\gamma = \frac{N+2}{N+1}$, and so $|\nabla u^n|$ is uniformly bounded in $L^s(Q)$ with $1 \leq s < \frac{N+2}{N+1}$.

Now, we shall use the above estimates to prove some *compactness* results that will be useful to pass to the limit in the distributional formulation for u^n. Indeed, thanks to these estimates, we can say that there exists a function $u \in L^q(0,1;W_0^{1,q}(\Omega))$, for all $q < \frac{N+2}{N+1}$, such that u^n converges to u weakly in $L^q(0,1;W_0^{1,q}(\Omega))$. On the other hand from the equation we deduce that $u_t^n \in L^1(Q) + L^{s'}(0,1;W^{-1,s'}(\Omega))$ uniformly with respect to n, where $s' = \frac{q}{p-1}$, for all $q < \frac{N+2}{N+1}$, and so, thanks to the *Simon's Theorem* (see Theorem 1.9) we have that u^n actually converges to u in $L^1(Q)$. All these facts allow us to pass to the limit in the distributional formulation of u^n and to conclude that u is a distributional solution of (4.34).

From Theorem 1.10 we get that u is also continuous with values in $L^1(\Omega)$, so that there are no problems to check that $u(0) = u_0$; indeed, we can multiply the equation solved by u by smooth functions which touch the level $\Omega \times \{0\}$ and comparing it with the problem solved by u^n.

□

4.1. Lack of uniqueness: Serrin's Counterexample

4.1.1. What happens in the elliptic case? Nothing has been said about uniqueness of distributional solutions of (2.1), which is still open, even in the elliptic framework with smooth data. In fact, in [**Se**], J. Serrin has shown that, if $N = 2$, and $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$, then for every fixed $0 < \varepsilon < 1$ there exists a matrix A^ε, such that

- $a_{i,j}^\varepsilon$ are measurabe functions defined on Ω, $\forall\ i,j = 1,2$,

- $a_{i,j}^{\varepsilon} \in L^{\infty}(\Omega),\ \forall\ i,j=1,2,$
- $A^{\varepsilon}(x)\ \xi\cdot\xi \geq \alpha_{\varepsilon}|\xi|^2$, for a.e. $x\in\Omega$, and for any $\xi\in\mathbb{R}^2$, with $\alpha>0$,

and

$$\begin{cases} u \in W_0^{1,q}(\Omega), & \forall\ 1 \leq q < \frac{2}{1+\varepsilon}, \\ \displaystyle\int_{\Omega} A^{\varepsilon}(x)\nabla u\cdot\nabla\varphi\ dx = 0, & \forall\varphi\in C_0^{\infty}(\Omega), \end{cases} \tag{4.42}$$

admits at least two solutions.

The Serrin's coefficients are

$$a_{i,j} = \left(\frac{1}{\varepsilon^2}-1\right)\frac{x_i x_j}{r^2} + \delta_{i,j}, \tag{4.43}$$

for $i,j=1,2.$, where $r=\sqrt{x_1^2+x_2^2}$ and $\delta_{i,j}$ stands for the Kronecker symbol; if $v(x)$ is the unique variational solution (see for instance [**E**]) of problem

$$\begin{cases} -\mathrm{div}(A^{\varepsilon}(x)\nabla v) = 0, & \text{in } \Omega, \\ v = x_1 & \text{on } \partial\Omega, \end{cases}$$

then $u = x_1 r^{-N+1-\varepsilon} - v(x)$ is a nontrivial (the trivial solution is obviously $u=0$) solution of problem (4.42). Let us notice that, this pathological solution found by Serrin belongs to $W_0^{1,q}(\Omega)$ for every $q\in[1,\frac{2}{1+\varepsilon})$, this is coherent with the uniqueness result we proved in Theorem 2.4.

In [**Pr1**], the author extended such a counterexample to the case $N\geq 3$. For instance, if $N=3$ the matrix is:

$$A^{\varepsilon} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $a_{i,j}$ are the same coefficients defined in (4.43).

4.1.2. Parabolic case. Unfortunately, as in the elliptic case, due to the lack of regularity of the solutions, the distributional formulation is not strong enough to provide uniqueness, as it can be checked by readapting to the parabolic case the counterexample of J. Serrin for the stationary problem.

Indeed let us fix $\frac{1}{2}<\varepsilon<1$; this way we have $\frac{2}{1+\varepsilon}<\frac{4}{3}$ (that is $\frac{2}{1+\varepsilon}<\frac{N+2}{N+1}$). Consider the Serrin's solution $u(x)$ to problem (4.42); it turns out to be as well a distributional solution of the associated parabolic problem

$$\begin{cases} w_t - \mathrm{div}(A_{\varepsilon}(x)\nabla w) = 0 \quad \text{in } Q \\ w(0) = u(x), \end{cases} \tag{4.44}$$

which also admits a solution $z(x,t)\in L^q(0,T;W_0^{1,q})$, for any $q<\frac{N+2}{N+1}$, as proved in Theorem 4.1. That is $z\neq u$ or, in other words, problem (4.44) admits two different distributional soltions.

4.2. Duality Approach: singular data

To overcome this problem Guido Stampacchia (see [**S**]) introduced, in the elliptic framework, a method to select the *right solution* for problem (4.34), that is, in the case of Serrin's pathology (4.42) the solution $u = 0$.

This notion starts from the clever idea to test the problem with smooth solutions of the *dual* problem. The argument is so powerful that allow us to prove existence of solutions (in this *duality sense*) even with very irregular data, namely measures.

Let us straightforwardly extend this definition to the parabolic case.

DEFINITION 4.2. Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$ A function $u \in L^1(Q)$ is a *duality solution* of problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t)\nabla u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.45}$$

if

$$-\int_\Omega u_0 w(0)\, dx + \int_Q u\, g\, dxdt = \int_Q f\, w\, dx, \tag{4.46}$$

for every $g \in L^\infty(Q)$, where w is the solution of the *backward problem*

$$\begin{cases} -w_t - \operatorname{div}(A^*(t,x)\nabla w) = g & \text{in } (0,T) \times \Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{4.47}$$

where $A^*(t,x)$ is the transposed matrix of $A(t,x)$.

REMARK 4.3. Notice that all terms in (4.46) are well defined thanks to Theorem 3.2. Moreover, it is quite easy to check that any duality solution of problem (4.45) actually turns out to be a distributional solution of the same problem. Finally recall that any duality solution turns out to coincide with the renormalized solution of the same problem (see [**Pe**]); this notion introduced in [**DMOP**] for the elliptic case, and then adapted to the parabolic case in [**Pe**], is the right one to ensure uniqueness also in the nonlinear framework. Finally notice that solutions of a *forward* parabolic problem and its associated *backward* problem are the same through the change of variable $t \mapsto -t$.

A unique duality solution for problem (4.45) exists, in fact we have the following

THEOREM 4.4. *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a unique duality solution of problem (4.45).*

PROOF. Let us fix $r, q \in \mathbb{R}$ such that

$$r,\, q > 1, \qquad \frac{N}{q} + \frac{2}{r} < 2\,,$$

and let us consider $g \in L^r(0,T;L^q(\Omega))$. Let w be the weak solution of problem (4.47); we know that w is bounded (Theorem 3.2) and continuous with values in $L^2(\Omega)$ (Theorem 2.4). We actually have

$$\|w\|_{L^\infty(Q)} \leq C\|g\|_{L^r(0,T;L^q(\Omega))};$$

therefore, the linear functional

$$\Lambda : L^r(0,T;L^q(\Omega)) \mapsto \mathbb{R},$$

defined by

$$\Lambda(g) = \int_Q f\, w\; dx + \int_\Omega u_0 w(0)\,,$$

is well-defined and continuous, since

$$|\Lambda(g)| \leq (\|f\|_{L^1(Q)} + \|u_0\|_{L^\infty(\Omega)})\|w\|_{L^\infty(Q)} \leq C\|g\|_{L^r(0,T;L^q(\Omega))}.$$

So, by *Riesz's representation theorem* there exists a unique u belonging to $L^{r'}(0,T;L^{q'}(\Omega))$ such that

$$\Lambda(g) = \int_Q u\, g\; dxdt,$$

for any $g \in L^r(0,T;L^q(\Omega))$. So we have that, if $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a (unique by construction) duality solution of problem (4.45). □

CHAPTER 5

Asymptotic behavior of the solutions

5.1. Naïve idea and main assumptions

Let us give a naïve idea of what happens to a solution for large times. Let $u(t,x)$ be the solution of the 1-D heat equation

$$\begin{cases} u_t - u_{x\,x} = 0 & \text{in } \ (0 < t < \infty) \times (0 < x < 1) \\ u(0,x) = u_0(x) & \text{on } \ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0 & \text{in } \ 0 \le t < \infty, \end{cases}$$

with smooth u_0 ($u_0(0) = u_0(1) = 0$). Since we can write the initial datum u_0 as the uniform convergent series

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

then the solution $u(t,x)$ is the explicitly given by

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin(n\pi x),$$

and so, $u(t,x)$ tends to zero (with exponential rate!) as $t \to \infty$. Let us just emphasize (actually, my mom too should be able to easily check it!) that $z(x) \equiv 0$ solves the associated elliptic Laplace equation

$$\begin{cases} -z_{x\,x} = 0 & \text{in } \ (0 < x < 1) \\ z(0) = z(1) = 0. \end{cases}$$

That is, the solution u tends to something constant (in time), and so its derivative with repect to t converges, in some sense, to zero.

A large number of papers has been devoted to the study of asymptotic behavior for solutions of parabolic problems under various assumptions and in different contexts: for a review on classical results see [**F**] and the references therein. More recently in [**Pe1**] and [**LP**] the case of nonlinear monotone operators, and quasilinear problems with nonlinear absorbing terms having natural growth, have been considered; in particular, in [**Pe1**], we dealt with nonnegative measures μ absolutely continuous with respect to the parabolic p-capacity (the so called *soft measures*). Here we analyze the case of linear operators with L^1 data and no sign assumptions on the data. We follow the outlines of [**Pe2**], where the result is proved in the setting of general, possibly singular, bounded measures. In fact, as we said, the

notion of duality solution is flexible enough to work as well for such singular data. Existence and uniqueness of a duality solution for measure data can be easily obtained by approximation using Theorem 4.4.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, $T > 0$; as usual, we denote by Q the cylinder $\Omega \times (0,T)$. We are interested in the study of the asymptotic behavior with respect to the time variable t of the solution of the linear parabolic problem

$$\begin{cases} u_t + L(u) = f & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{5.48}$$

with $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and

$$L(u) = -\mathrm{div}(A(x)\nabla u),$$

where A is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{5.49}$$

for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$.

We just want to treat the simpler case of a duality solution of problem (5.48), in the case where f does not depend on time, even if slight generalizations are possible.

First observe that by Theorem 4.4 a unique solution is well defined for all $t > 0$. We recall (see [**S**]) that by a duality solution of problem

$$\begin{cases} L(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.50}$$

we mean a function $v \in L^1(\Omega)$ such that

$$\int_\Omega v\, g\, dx = \int_\Omega f z\, dx, \tag{5.51}$$

for every $g \in L^\infty(\Omega)$, where z is the variational solution of the dual problem

$$\begin{cases} -\mathrm{div}(A^*(x)\nabla z) = g & \text{in } \Omega, \\ z(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.52}$$

Thanks to Theorem 1.10, a duality solution of problem (5.48) turns out to be continuous with values in $L^1(\Omega)$.

Let us state our main result:

THEOREM 5.1. *Let $f \in L^1(Q)$ be independent on the variable t. Let $u(x,t)$ be the duality solution of problem (5.48) with $u_0 \in L^1(\Omega)$, and let $v(x)$ be the duality solution of the corresponding elliptic problem (5.50). Then*

$$\lim_{T\to+\infty} u(x,t) = v(x),$$

in $L^1(\Omega)$.

5.2. Asymptotic behavior

In this section we will prove Theorem 5.1. Let us first prove the following preliminary result:

PROPOSITION 5.2. *Let $f \in L^1(Q)$ be independent on time and let v be the duality solution of the elliptic problem (5.50). Then v is the unique solution of the parabolic problem (5.48), with $u_0 = v$, in the duality sense introduced in Definition 4.2, for any fixed $T > 0$.*

PROOF. We have to check that v is a solution of problem (5.48); to do that let us choose $T_k(v)$ as test function in (4.47). We obtain

$$-\int_0^T \langle w_t, T_k(v)\rangle \; dt + \int_Q A^*(x)\nabla w \cdot \nabla T_k(v) \; dxdt = \int_Q T_k(v) \; g \; dxdt.$$

Now, integrating by parts we have

$$-\int_0^T \langle w_t, T_k(v)\rangle \; dt = \int_\Omega w(0)v(x) + \omega(k),$$

where $\omega(k)$ denotes a nonnegative quantity which vanishes as k diverges, while

$$\int_Q T_k(v) \; g \; dxdt = \int_Q v \, g \; dxdt + \omega(k).$$

Finally, using Theorem 2.33 and Theorem 10.1 of [**DMOP**], we have

$$\int_Q A^*(x)\nabla w \cdot \nabla T_k(v) \; dxdt = \int_Q A(x)\nabla T_k(v) \cdot \nabla w \; dxdt$$

$$= \int_0^T \int_\Omega f_k w \; dx \, dt,$$

where $f_k = f\chi_{\{v<k\}} + \mu_k$, with μ_k a sequence of measures converging to zero; thus, recalling that w is bounded, we have

$$\int_Q A^*(x)\nabla w \cdot \nabla T_k(v) \; dxdt = \int_Q fw \; dx + \omega(k).$$

Gathering together all these facts, we have that v is a duality solution of (5.48) having itself as initial datum. □

Let us give the following definition:

DEFINITION 5.3. A function $u \in L^1(Q)$ is a *duality supersolution* of problem (5.48) if

$$\int_Q u \, g \; dxdt \geq \int_Q fw \; dxdt + \int_\Omega u_0 w(0) \; dx,$$

for any bounded $g \geq 0$, and w solution of (4.47), while u is a *duality subsolution* if $-u$ is a duality supersolution.

By linearity we easily deduce

LEMMA 5.4. *Let $\overline{u}$ and $\underline{u}$ be respectively a duality supersolution and a duality subsolution for problem (5.48). Then $\underline{u} \le \overline{u}$.*

Observe that, if the functions in Lemma 5.4 are continuous with values in $L^1(\Omega)$, then we actually have that $\underline{u}(x,t) \le \overline{u}(x,t)$ for every fixed t, a.e on Ω.

PROOF OF THEOREM 5.1. We split the proof in few steps.

Step 1. Let us first suppose $u_0 = 0$ and $f \ge 0$. If we consider a parameter $s > 0$ we have that both $u(x,t)$ and $u_s(x,t) \equiv u(x,t+s)$ are duality solutions of problem (5.48) with, respectively, 0 and $u(x,s) \ge 0$ as initial datum; so, from Lemma 5.4 we deduce that $u(x,t+s) \ge u(x,t)$ for $t, s > 0$. Therefore u is a monotone nondecreasing function in t and so it converges to a function $\tilde{v}(x)$ almost everywhere and in $L^1(\Omega)$ since, thanks to Proposition 5.2 and Lemma 5.4, $u(x,t) \le v(x)$.

Now, let us consider $u^n(x,t)$ as the solution of

$$\begin{cases} (u^n)_t - \operatorname{div}(A(x)\nabla u^n) = f & \text{in } \Omega \times (0,1), \\ u^n(x,0) = u(x,n) & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega \times (0,1). \end{cases} \tag{5.53}$$

On the other hand, if $g \ge 0$, we define $w^n(x,t)$ as

$$\begin{cases} -w^n_t - \operatorname{div}(A^*(x)\nabla w^n) = g & \text{in } \Omega \times (0,1), \\ w^n(x,1) = w(x,n+1) & \text{in } \Omega, \\ w^n = 0 & \text{on } \partial\Omega \times (0,1). \end{cases} \tag{5.54}$$

Recall that, through the change of variable $s = T - t$, w solves a related linear parabolic problem, so that if $g \ge 0$, by classical comparison results one has that $w(x,t)$ is decreasing in time. Moreover, by comparison principle, we have that w^n is increasing with respect to n and, again by comparison Lemma 5.4, we have that, for fixed $t \in (0,1)$

$$w^n(x,1) \le w^n(x,t) = w(x,n+t) \le w(x,n) = w^{n-1}(x,1),$$

and so its limit $\tilde{w}$ does not depend on time and is the solution of elliptic dual problem (5.52). An analogous argument shows that also the limit of u^n does not depend on time. Thus, using u^n in (5.54) and w^n in (5.53), integrating by parts and subtracting, we obtain

$$\int_0^1 \int_\Omega u^n\, g - \int_0^1 \int_\Omega f w^n \, dx + \int_\Omega u^n(0) w^n(0) \, dx - \int_\Omega u^n(1) w^n(1) \, dx = 0.$$

Hence, we can pass to the limit on n using monotone convergence theorem obtaining

$$\int_\Omega \tilde{v}\, g - \int_\Omega f \tilde{w} = 0, \tag{5.55}$$

and so $v = \tilde{v}$.

If g has no sign we can reason separately with g^+ and g^- obtaining (5.55) and then using the linearity of (4.46) to conclude.

If v is the duality solution of problem (5.50), we proved in Proposition 5.2 that v is also the duality solution of the initial boundary value problem (5.48) with v itself as initial datum. Therefore, by comparison Lemma 5.4, if $0 \leq u_0 \leq v$, we have that

$$u(x,t) \to v \quad \text{in} \quad L^1(\Omega),$$

as t tends to infinity; in fact, we proved it for the duality solution with homogeneous initial datum, while v is a nonnegative duality solution with itself as initial datum.

Step 2. Now, let us take $u_\lambda(x,t)$ the solution of problem (5.48) with $u_0 = \lambda v$ as initial datum for some $\lambda > 1$ and again $f \geq 0$. Hence, since λv does not depend on time, we have that it is a duality supersolution of the parabolic problem (5.48), and, observing that v is a subsolution of the same problem, we can apply again the comparison lemma finding that

$$v(x) \leq u_\lambda(x,t) \leq \lambda v(x)$$

a.e. in Ω, for all positive t.

Moreover, thanks to the fact that the datum f does not depend on time, we can apply the comparison result also between $u_\lambda(x, t+s)$ solution with $u_0 = u_\lambda(x,s)$, with s a positive parameter, and $u_\lambda(x,t)$, the solution with $u_0 = \lambda v$ as initial datum; so we obtain

$$u_\lambda(x, t+s) \leq u_\lambda(x,t)$$

for all $t, s > 0$, a.e. in Ω. So, by virtue of this monotonicity result we have that there exists a function $\overline{v} \geq v$ such that $u_\lambda(x,t)$ converges to $\overline{v}$ a.e. in Ω as t tends to infinity. Clearly $\overline{v}$ does not depend on t and we can develop the same argument used before to prove that we can pass to the limit in the approximating duality formulation, and so, by uniqueness, we can obtain that $\overline{v} = v$. So, we have proved that the result holds for the solution starting from $u_0 = \lambda v$ as initial datum, with $\lambda > 1$ and $f \geq 0$. Since we proved before that the result holds true also for the solution starting from $u_0 = 0$, then, again applying a comparison argument, we can conclude in the same way that the convergence to v holds true for solutions starting from u_0 such that $0 \leq u_0 \leq \lambda v$ as initial datum, for fixed $\lambda > 1$.

Step 3. Now, let $u_0 \in L^1(\Omega)$ a nonnegative function and $f \geq 0$, and recall that, thanks to suitable Harnack inequality (see [**T**]), if $f \neq 0$, then $v > 0$ (which implies λv tends to $+\infty$ on Ω as λ diverges). Without loss of generality we can suppose $f \neq 0$ (the case $f \equiv 0$ is the easier one and it can be proved, for instance, as in [**Pe1**]); let us define the monotone nondecreasing (with respect to λ) family of functions

$$u_{0,\lambda} = \min(u_0, \lambda v).$$

As we have shown above, for every fixed $\lambda > 1$, $u_\lambda(x,t)$, the duality solution of problem (5.48) with $u_{0,\lambda}$ as initial datum, converges to v a.e.

in Ω, as t tends to infinity. Moreover, using again standard compactness arguments,we also have that

$$T_k(u_\lambda(x,t)) \rightharpoonup T_k(v)$$

weakly in $H_0^1(\Omega)$ as t diverges, for every fixed $k > 0$.

So, thanks to *Lebesgue theorem*, we can easily check that $u_{0,\lambda}$ converges to u_0 in $L^1(\Omega)$ as λ tends to infinity. Therefore, using a stability result for renormalized solutions of the linear problem (5.48) (see [**Pe**]) we obtain that $T_k(u_\lambda(x,t))$ converges to $T_k(u(x,t))$ strongly in $L^2(0,T;H_0^1(\Omega))$ as λ tends to infinity.

On the other hand, since $z_\lambda = u - u_\lambda$ solves the problem (5.48) with $u_0 - u_{0,\lambda}$ as initial datum, then z_λ turns out to be an entropy solution of the same problem and so we have (see [**Pr2**])

$$\int_\Omega \Theta_k(u - u_\lambda)(t)\ dx \le \int_\Omega \Theta_k(u_0 - u_{0,\lambda})\ dx,$$

for every $k,\ t > 0$. Dividing the above inequality by k, and passing to the limit as k tends to 0 we obtain

$$\|u(x,t) - u_\lambda(x,t)\|_{L^1(\Omega)} \le \|u_0(x) - u_{0,\lambda}(x)\|_{L^1(\Omega)}, \tag{5.56}$$

for every $t > 0$. Hence, we have

$$\|u(x,t) - v(x)\|_{L^1(\Omega)} \le \|u(x,t) - u_\lambda(x,t)\|_{L^1(\Omega)} + \|u_\lambda(x,t) - v(x)\|_{L^1(\Omega)};$$

then, thanks to the fact that the estimate in (5.56) is uniform in t, for every fixed ϵ, we can choose $\bar\lambda$ large enough such that

$$\|u(x,t) - u_{\bar\lambda}(x,t)\|_{L^1(\Omega)} \le \frac{\epsilon}{2},$$

for every $t > 0$; on the other hand, thanks to the result proved above, there exists $\bar t$ such that

$$\|u_{\bar\lambda}(x,t) - v(x)\|_{L^1(\Omega)} \le \frac{\epsilon}{2},$$

for every $t > \bar t$, and this concludes the proof of the result in the case of nonnegative data f and $u_0 \in L^1(\Omega)$.

Step 4. Let $f \in L^1(Q)$ be independent on t and $u_0 \in L^1(\Omega)$ with no sign assumptions. We consider the function $z(x,t) = u(x,t) - v(x)$; thanks to Proposition 5.2 it turns out to solve problem (5.48) with $u_0 - v$ as initial data and $f = 0$, and so, if either $u_0 \le v$ or $u_0 \ge v$ then the result is true since $z(x,t)$ tends to zero in $L^1(\Omega)$ as t diverges thanks to what we proved above. Now, if $u^\oplus$ and $u^\ominus$ solve problem (5.48) with, respectively, $\max(u_0, v)$ and $\min(u_0, v)$ as intital data, then, by comparison, we have

$$u^\ominus(x,t) \le u(x,t) \le u^\oplus(x,t)$$

for any t, a. e. in Ω, and this concludes the proof since the result holds true for both $u^\oplus$ and $u^\ominus$.

□

APPENDIX A

Basic tools in integration and measure theory

We set by $\mathbb{R}^N$ the N-Euclidian space (simply $\mathbb{R}$ if $N = 1$) on which the standard Lebesgue measure is defined on the σ-algebra of Lebesgue measurable sets. The scalar product between two vectors a, b in $\mathbb{R}^N$ will be denoted by $a \cdot b$ or simply ab in most cases. Given a bounded open set Ω of $\mathbb{R}^N$, whose boundary will be denoted by $\partial\Omega$, and given a positive T, we shall consider the cylinder $Q_T = (0,T) \times \Omega$ (or simply Q where there is no possibility of confusion), setting by $C_0(Q)$ and $C_0^\infty(Q)$, the space of continuous, respectively C^∞, functions with compact support in Ω, while $C(\overline{\Omega})$ will denote functions that are continuous in the whole closed set $\overline{\Omega}$; moreover we will indicate by $C_0^\infty([0,T] \times \Omega)$ (resp. $C_0^\infty([0,T) \times \Omega)$) the set of all C^∞ functions with compact support on the set $[0,T] \times \Omega)$ (resp. on $[0,T) \times \Omega$).

For the sake of simplicity here we will denote by D any bounded open subset of $\mathbb{R}^N$. We will deal with the space $M(D)$ of Radon measures μ on D that, by means of *Riesz's representation theorem*, turns out to coincide with the dual space of $C_0(D)$ with the topology of locally uniform convergence; we shall identify the element μ in $M(D)$ with the real valued additive set function associated, which is defined on the σ-algebra of Borel subsets of D and is finite on compact subsets. Thus with μ^+ and μ^- we mean, respectively, the positive and the negative variation of the Hahn decomposition of μ, that is $\mu = \mu^+ - \mu^-$, while the total variation of μ will be denoted by $|\mu| = \mu^+ + \mu^-$. Since we will always deal with the subset of $M(D)$ of the measures with bounded total variation on D, to simplify the notation we will denote also by $M(D)$ this subset. The restriction of a measure μ on a subset E is denoted by $\mu \llcorner E$ and is defined as follows:

$$(\mu \llcorner E)(B) = \mu(E \cap B), \quad \text{for every Borel subset } B \subseteq D.$$

If $\mu = \mu \llcorner E$ we will say that μ is concentrated on E.

For $1 \leq p \leq \infty$, we denote by $L^p(D)$ the space of Lebesgue measurable functions (in fact, equivalence classes, since almost everywhere equal functions are identified) $u : D \to \mathbb{R}$ such that, if $p < \infty$

$$\|u\|_{L^p(D)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

or which are essentially bounded (w.r.t. Lebesgue measure) if $p = \infty$. For the definition, the main properties and results on Lebesgue spaces we refer

to [**B**]. For a function u in a Lebesgue space we set by $\dfrac{\partial u}{\partial x_i}$ (or simply u_{x_i}) its partial derivative in the direction x_i defined in the sense of distributions, that is

$$\langle u_{x_i}, \varphi \rangle = -\int_D u\varphi_{x_i}\, dx,$$

and we denote by $\nabla u = (u_{x_1}, \dots, u_{x_N})$ the gradient of u defined this way.

The Sobolev space $W^{1,p}(D)$ with $1 \le p \le \infty$, is the space of functions u in $L^p(D)$ such that $\nabla u \in (L^p(D))^N$, endowed with its natural norm $\|u\|_{W^{1,p}(D)} = \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, while $W_0^{1,p}(D)$ will indicate the closure of $C_0^\infty(D)$ with respect to this norm. We still follow [**B**] for basic results on Sobolev spaces. Let us just recall that, for $1 < p < \infty$, the dual space of $L^p(D)$ can be identified with $L^{p'}(D)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent of p, and that the dual space of $W_0^{1,p}(D)$ is denoted by $W^{-1,p'}(D)$. By a well known result, any element of $T \in W^{-1,p'}(D)$ can be written in the form $T = -\mathrm{div}(G)$ where $G \in (L^{p'}(D))^N$.

For every $0 < p < \infty$, we introduce the Marcinkiewicz space $M^p(D)$ of measurable functions f such that there exists $c > 0$, with

$$\mathrm{meas}\{x : |f(x)| \ge k\} \le \frac{c}{k^p},$$

for every positive k; it turns out to be a Banach space endowed with the pseudo-norm

$$\|f\|_{M^p(D)} = \inf\left\{c > 0 : \mathrm{meas}\{x : |f(x)| \ge k\} \le \left(\frac{c}{k}\right)^p\right\}.$$

Let us recall that, since D is bounded, then for $p > 1$ we have the following continuous embeddings

$$L^p(D) \hookrightarrow M^p(D) \hookrightarrow L^{p-\varepsilon}(D),$$

for every $\varepsilon \in (0, p-1]$.

We already said that we refer to [**B**] for most basic tools in Lebesgue theory and Sobolev spaces; however, among them, let us recall explicitly some that will play a crucial role in the methods we use.

(1) *Generalized Young's inequality:* for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ε we have:

$$ab \le \varepsilon^p \frac{a^p}{p} + \frac{1}{\varepsilon^{p'}} \frac{b^{p'}}{p'}, \qquad \forall a, b > 0.$$

(2) *Hölder's inequality:* for $1 < p < \infty$, $p' = \frac{p}{p-1}$, we have, for every $f \in L^p(D)$ and every $g \in L^{p'}(D)$:

$$\int_D |fg|\, dx \le \left(\int_D |f|^p\right)^{\frac{1}{p}} \left(\int_D |g|^{p'}\right)^{\frac{1}{p'}}.$$

(3) Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $\{f_n\} \subset L^p(D)$, $\{g_n\} \subset L^{p'}(D)$ be such that f_n strongly converges to f in $L^p(D)$ and g_n weakly converges to g in $L^{p'}(D)$. Then

$$\lim_{n\to\infty} \int_D f_n\, g_n\, dx = \int_D f g\, dx\,.$$

The same conclusion holds true if $p = 1$, $p' = \infty$ and the weak convergence of g_n is replaced by the $*$-weak convergence in $L^\infty(D)$. Moreover, if f_n strongly converges to zero in $L^p(D)$, and g_n is bounded in $L^{p'}(D)$, we also have

$$\lim_{n\to\infty} \int_D f_n\, g_n\, dx = 0\,.$$

(4) Let f_n converge to f in measure and suppose that:

$$\exists\, C > 0, \quad q > 1 : \quad \|f_n\|_{L^q(D)} \le C, \quad \forall\, n.$$

Then

$$f_n \longrightarrow f \quad \text{strongly in } L^s(D), \text{ for every } 1 \le s < q.$$

(5) *Fatou's lemma:* Let $\{f_n\} \subset L^1(D)$ be a sequence such that $f_n \to f$ a.e. in D and $f_n \ge h(x)$ with $h(x) \in L^1(D)$, then

$$\int_D f\, dx \le \liminf_{n\to\infty} \int_D f_n\, dx.$$

(6) *Generalized Lebesgue theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and $|f_n| \le g_n$ with g_n strongly convergent in $L^p(D)$, then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(7) Let $\{f_n\} \subset L^1(D)$ and $f \in L^1(D)$ be such that, $f_n \ge 0$, $f_n \to f$ a.e. in D, and

$$\lim_{n\to\infty} \int_D f_n\, dx = \int_D f\, dx,$$

then f_n strongly converges to f in $L^1(D)$.

(8) *Vitali's theorem:* Let $1 \le p < \infty$, and let $\{f_n\} \subset L^p(D)$ be a sequence such that $f_n \to f$ a.e. in D and

$$\lim_{\mathrm{meas}(E)\to 0} \sup_n \int_E |f_n|^p\, dx = 0. \tag{A.57}$$

Then $f \in L^p(D)$ and f_n strongly converges to f in $L^p(D)$.

(9) Let $\{f_n\} \subset L^1(D)$ and $\{g_n\} \subset L^\infty(D)$ be two sequences such that

$$f_n \longrightarrow f \quad \text{weakly in } L^1(D),$$

$$g_n \longrightarrow g \quad \text{a.e. in } D \text{ and } *\text{-weakly in } L^\infty(D).$$

Then

$$\lim_{n\to\infty} \int_D f_n\, g_n\, dx = \int_D f g\, dx\,.$$

REMARK A.1. Property (A.57) is the so called equi-integrability property of the sequence $\{|f_n|^p\}$. We recall that *Dunford-Pettis theorem* ensures that a sequence in $L^1(D)$ is weakly convergent in $L^1(D)$ if and only if it is equi-integrable. Moreover, results (4), (6) and (7) can be proven as an easy consequences of Vitali's theorem and so we will refer to them as Vitali's theorem as well. For the same reason we will refer to result (9) as *Egorov theorem*.

For functions in the Sobolev space $W_0^{1,p}(D)$ we will often use *Sobolev's theorem* stating that, if $p < N$, $W_0^{1,p}(D)$ continuously injects into $L^{p^*}(D)$ with $p^* = \frac{Np}{N-p}$; if $p = N$, $W_0^{1,p}(D)$ continuously injects into $L^q(D)$ for every $q < \infty$, while, if $p > N$, $W_0^{1,p}(D)$ continuously injects into $C(\overline{D})$. Let us also recall *Rellich's theorem* stating that, if $p < N$, the injection of $W_0^{1,p}(D)$ into $L^q(D)$ is compact for every $1 \le q < p^*$, and *Poincaré's inequality*, that is, there exists $C > 0$ such that

$$\|u\|_{L^p(D)} \le C\|\nabla u\|_{(L^p(D))^N},$$

for every $u \in W_0^{1,p}(D)$, so that $\|\nabla u\|_{(L^p(D))^N}$ can be used as equivalent norm on $W_0^{1,p}(D)$.

We will often use the following result due to G. Stampacchia.

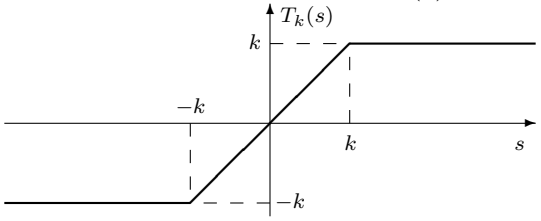
THEOREM A.2. *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(D)$ we have $G(u) \in W_0^{1,p}(D)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in D.*

PROOF. See [**S**]. □

Theorem A.2 has an important consequence, that is

$$\nabla u = 0 \quad \text{a.e. in } F_c = \{x : u(x) = c\},$$

for every $c > 0$. Hence, we are able to consider the composition of function in $W_0^{1,p}(D)$ with some useful auxiliary function. One of the most used will be the truncation function at level $k > 0$, that is $T_k(s) = \max(-k, \min(k, s))$;



thus, if $u \in W_0^{1,p}(D)$, we have that $T_k(u) \in W_0^{1,p}(D)$ and $\nabla T_k(u) = \nabla u \chi_{\{u<k\}}$ a.e. on D, for every $k > 0$.

If u is such that its truncation belongs to $W_0^{1,p}(D)$, then we can define an *approximated gradient* of u defined as the a.e. unique measurable function $v : D \to \mathbb{R}^N$ such that

$$v = \nabla T_k(u) \tag{A.58}$$

almost everywhere on the set $\{|u| \leq k\}$, for every $k > 0$ (see for instance [**B6**])

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