Universidad de Granada

NONLINEAR ANALYSIS AND DIFFERENTIAL EQUATIONS[1]

Notes on the course **Nonlinear analysis and differential equations**, from the master's degree FisyMat, due to a realization of a collaboration grant with the Department of Mathematical Analysis of the University of Granada

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Note for the reader

The present work constitutes a complete notes on the FisyMat-course *Nonlinear analysis and differential equations* carried out in 2018-2019 and it might contain typos, also can be subjected to changes or improvements. The work is divided into three theoretical chapters named *The topological method*, *Sobolev spaces* and *The variational method*, each one with their corresponding exercises whose solutions are exposed in Spanish at the end of the work.

Contents

Chapter 1

The topological method

The topological method is just an astonishing topological tool that permits to solve systems of equations in general. The chapter is divided into six sections (three devoted to Brouwer degree and three devoted to Leray-Schauder degree). The main bibliography used here is [1] and [7] for the first part of this chapter and [6] and [12] for the second one.

1.1 Introduction and motivation for the Brouwer degree

The problem of solving a general equation is really significant not only in Mathematics but also in disciplines like Physics, Engineering or Economics, among others and arise in a large number of applications in daily life. Mathematicians have dealt with the ancient problem of solving equations all along the history although, unfortunately, it is impossible to solve explicitly a general equation because it heavily depends on the type of equation. Here, some examples are presented:

Example 1.

1. $2^{x-1} + 2^x + 2^{x+1} = 28$
   This is an example of an expotential equation with base two which can be solved by making the variable change $y = 2^x$ *or, if it is preferred, by taking the common factor* 2^{x-1} *and writing* $2^{x-1}(1+2+4) = 28$ *or, equivalently,* $2^{x-1} = 2^2$ *and then using the injectivity of the exponential function with base two to conclude that the solution is* $x = 3$*. Note that there is unicity of solution and it is explicitly calculable.*

2. $x^4 - 5x^2 + 6 = 0$
   This is an example of a biquadratic equation which can be solved by making the variable change $y = x^2$ *and then applying the famous formula for the solutions of a quadratic equation to conclude that the solutions are* $x = \pm\sqrt{2}$ *and* $x = \pm\sqrt{3}$*. Note that there is no unicity of solution, but there are a finite number of them, and they all are explicitly calculable.*

3. $4\sin(x) - \cos(2x) + 1 = 0$
This is an example of a trigonometric equation which can be solved by applying the cosine double-angle formula and the fact that $\cos^2(x) + \sin^2(x) = 1$ *for all* $x \in \mathbb{R}$ *to reduce it to the easier equation* $\sin(x)[\sin(x) - 2] = 0$ *which obviously is equivalent to* $\sin(x) = 0$, *so the solutions are of the form* $x_k = k\pi$ *for all* $k \in \mathbb{Z}$. *Note that there is no unicity of solution, moreover there are an infinite number of them, and they all are still explicitly calculable.*

All the equations from the above example can be solved explicitly using basic calculus and algebra. Nevertheless, this is not always the case: too many equations that apparently have a simple expression cannot be solved explicitly, for example:

- $x^5 - 5x - 1 = 0$

From the theory of Galois, it is known that polynomial equations of degree greater than four cannot be solved explicitly in general. The equation $x^5 - 5x - 1 = 0$ is an example of a polynomial equation of degree five and the classical method for finding integer or rational roots of a polynomial equation, meaning the Ruffini method, does not work successfully here but the reality is that the equation has three real solutions, as it is shown by its graph:

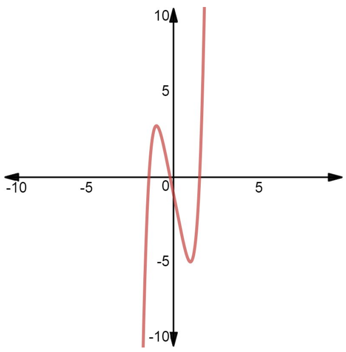

Figure 1.1: Graph of $x^5 - 5x - 1$

- $e^x + x^3 + x + \cos(x) = \kappa$

This equation mixes different elementary functions (exponential, potential and trigonometric ones) and it has only one real solution for each $\kappa \in \mathbb{R}$, as it is shown by its graph:

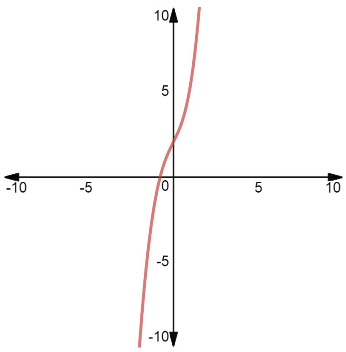

Figure 1.2: Graph of $e^x + x^3 + x + \cos(x)$

How is it possible to obtain the existence of solution in the previous examples? The answer is in the well-known *Bolzano's theorem* that assures the existence of solution in these more complicated situations. Futhermore, an additional study of the monotonicity of the given function together with this theorem, provides a complete study of existence and multiplicity of the considered equation.

Bolzano's theorem contains all the conditions of a very good theorem: simple statement, affordable proof and a large and wide applicability in the scientific world. Bolzano published in 1817 a paper under the title *Purely analytic proof of the theorem that between any two values which give results of opposite sign there lies at least one real root of the equation* [2], the first analytical proof of that result known today as the Bolzano's theorem. Despite this result was already known before Bolzano, there was no rigurous proofs of it: all the existing proofs before Bolzano used to be based on geometrical arguments. Actually, in these lines one can appreciate the mathematical rigour that Bolzano was looking for (see pages 160 and 161 of [2]):

> *But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*
>
> *On the other hand, we strictly require only this: that examples never be put forward instead of proofs and that the essence of a deduction never be based on the merely metaphorical use of phrases or on their related ideas, so that the deduction itself would become void as soon as these were changed.*

For a complete biography of the Czech mathematician Bernard Bolzano, see [4]. In his famous paper of 1817, Bolzano, in addition to exposed a purely analytical proof of the result he, at the same time, criticised the previous proofs of the theorem because of the aboundance of geometrical considerations. Among others relevant things, Bolzano gave a formal definition of the notion of continuity and established the so-called supremum and infimum existence theorem for a non-empty bounded set of real numbers. With nothing more to add, here is the statement of the Bolzano's theorem whose proof is left as an exercise.

Theorem 1 (Bolzano, 1817)**.** *Let $a, b \in \mathbb{R}$ be two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ a real-valued continuous function defined on the real interval $[a, b]$ such that $f(a)f(b) < 0$, then there exists $c \in]a, b[$ with $f(c) = 0$ or, equivalently, the equation $f(x) = 0$ has a solution in $]a, b[$.*

Exercise 1. *Prove the Bolzano's theorem.*

Equipped with this tool and the monotonicity of the considered function, now it is easy to prove the existence and unicity of solution of the equation $x^5 - 5x - 1 = 0$ and $e^x + x^3 + x + \cos(x) = a$ for all $a \in \mathbb{R}$. For example, in the case of the equation $x^5 - 5x - 1 = 0$, just define the derivable function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 - 5x - 1$ for all $x \in \mathbb{R}$. Clearly,

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

which gives the existence of two real numbers $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) < 0$ and $f(b) > 0$, so the Bolzano's theorem guarantee the existence of solution of the equation $f(x) = 0$ or if it is preferable, take $a = -1$ and $b = 1$ and then apply the Bolzano's theorem to the considered function f. In order to study the multiplicity of solutions, one computes its derivative f',

$$f'(x) = 5x^4 - 5 \quad \forall x \in \mathbb{R}$$

which vanishes at the points $+1$ and -1. Since $f'(x) > 0$ for each $x < 1$, $f'(x) < 0$ for each $x \in]-1, 1[$ and finally $f'(x) > 0$ for each $x > 1$, it follows that f is strictly increasing in $]-\infty, 1[\cup]1, +\infty[$ and strictly decreasing in $]-1, 1[$. In summary, one ends up with the fact that $x = -1$ is a local maximum for f with value $f(-1) = 3$ and $x = 1$ is a local minimum for f with value $f(1) = -5$ (see Figure 1.1). From the previous calculus, one is able to sketch the graph of f and conclude that there are exactly three real solutions to the equation $f(x) = 0$.

As another example, if the function $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^m + h(x)$ for all $x \in \mathbb{R}$, where $m \in \mathbb{N}$ is odd and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{|x| \to +\infty} \frac{h(x)}{|x|^m} = 0$, then the equation $g(x) = 0$ has a solution. This is the case of a polynomial equation of odd degree, as well as the case where m is odd and the function h is continuous and bounded.

Exercise 2. *Prove the existence and unicity of the equation $e^x + x^3 + x + \cos(x) = a$ for all $a \in \mathbb{R}$. If $g : \mathbb{R} \to \mathbb{R}$ is the function that maps a real number a to the unique solution x_a of $e^x + x^3 + x + \cos(x) = a$, is g continuous?.*

Once the existance of solution is proved, one can located the solution as much as wanted by using the bisection method, for example. As a direct consequence of the Bolzano's Theorem, it is obtained the *intermediate value theorem*.

Corollary 1 (Intermediate value)**.** *Let $a, b \in \mathbb{R}$ be two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ a real-valued continuous function defined on the real interval $[a, b]$, then the image $f([a, b])$ of f is a real interval.*

The Bolzano's theorem admits a version for scalar equations with several variables which can be easily prove by using the theorem 1.

Corollary 2. *Let $n \in \mathbb{N}$ be a natural number, $\Omega \subset \mathbb{R}^n$ a nonempty convex subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a real-valued continuous function defined on Ω such that there exists two points $a, b \in \Omega$ with $f(a)f(b) < 0$, then there exists $c \in (a, b)$ with $f(c) = 0$.*

Remark 1. *In the previous corollary, the notation $]a, b[$ with $a, b \in \mathbb{R}^n$ means the open segment of $\mathbb{R}^n$ defined by $]a, b[= \{(1 - \lambda)a + \lambda b : \lambda \in]0, 1[\}$.*

Exercise 3. *Prove the corollary 2.*

Example 2. *If the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded, then the equation $xe^y + h(x, y, z) = k$ has solution for each $k \in \mathbb{R}$ and this follows from the fact that*

$$\lim_{x \to +\infty} xe^y + h(x, y, z) = +\infty, \quad \forall y, z \in \mathbb{R}$$

$$\lim_{x\to-\infty} xe^y + h(x,y,z) = -\infty, \quad \forall y,z \in \mathbb{R}$$

For example, this is the case of the equation

$$xe^y + x^2e^{-x^2} + \sin(xy^5 + \log(1+x^2)) = 0$$

On the one hand, the analytical properties that allow to prove the Bolzano's theorem are basically two: the supremum existence theorem (something that makes no sense in several variables) and the locally sign conservation for continuous and nonzero functions (something that makes no sense for functions of several components). These properties are characteristic of the set of real numbers, so the reader could wonder if the Bolzano's theorem is still true or false when the set of rational numbers plays a role.

Exercise 4. *Prove or disprove the next situation: a continuous function $h : \mathbb{Q} \to \mathbb{R}$ such that there exist $a, b \in \mathbb{Q}$ with $h(a)h(b) < 0$, but there is no $x \in \mathbb{Q}$ with $h(x) = 0$.*

On the other hand, the fundamental properties that allow to prove the Bolzano's theorem are also two: the fact that the connectedness is a topological invariant, wich means that continuous functions map connected sets into connected sets and the fact that a non-trivial set of $\mathbb{R}$ is connected if, and only if, is an interval (and obviously this last property has not translation into greater dimensions). So, at this point, one should note that considering not only the existence but also the multiplicity of solutions, the situation may be completely different from the scalar case. In the following, this statement is clarified.

If the function $f : I \to \mathbb{R}$ is of class $\mathcal{C}^1$ on a non-trivial interval I of $\mathbb{R}$ with $f'(x) \neq 0$ for all $x \in I$ and the equation $f(x) = 0$ has solution, then the solution is unique because $f' > 0$ or $f' < 0$ in all the interval I (this statement holdsdue to the intermediate value theorem applied to f'). However, the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x+y$ for all $(x,y) \in \mathbb{R}^2$ is of class $\mathcal{C}^1$ on $\mathbb{R}^2$ and has infinitely many solutions, although both partial derivaties are positive in $\mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 1 > 0 \quad \forall (x,y) \in \mathbb{R}^2$$

This example must not be surprising, since if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function then its derivative f' is a function defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the sign of the vector $f'(x)$ cannot be defined in a appropiate way.

The situation is much more complicated in the case of systems ofo equations of the form $f(x) = 0$ with $f = (f_1, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. For instance, the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = ((e^y+1)\sin x, (e^y+1)\cos x)$ for all $(x,y) \in \mathbb{R}^2$ is continuous and its image contains points in the four quadrants of $\mathbb{R}^2$, but the equation $f(x,y) = (0,0)$ has no solutions, since $f(\mathbb{R}^2) = \mathbb{R}^2 \setminus \overline{B}_{\mathbb{R}^2}(0,1)$.

Exercise 5. *Prove that $f(\mathbb{R}^2) = \mathbb{R}^2 \setminus \overline{B}_{\mathbb{R}^2}(0,1)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x,y) = ((e^y+1)\sin x, (e^y+1)\cos x)$ for all $(x,y) \in \mathbb{R}^2$.*

Apart from the continuity of the considered function, the main hypothesis in Bolzano's theorem is that the image of the function takes values into the two sets $\mathbb{R}^+$ and $\mathbb{R}^-$, but it is clear from the previous example that the key idea to study systems of equations is not that the image of the function takes values into the 2^n subsets

$$\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0, ..., x_n > 0\}$$

$$\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 < 0, x_2 > 0, ..., x_n > 0\}$$

$$\vdots$$

$$\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 < 0, x_2 < 0, ..., x_n < 0\}$$

Just to prove the existence of solutions for systems equations the key idea is to demand a suitable behaviour at the topological boundary of the domain of the considered function. Turning to Bolzano's theorem, its hypothesis are given in terms of the behaviour of the continuous function f on the topological boundary of $[a, b]$, specifically $\{a, b\}$.

The most simple generalization of Bolzano's theorem is to consider $n = 2$, $\Omega =]a, b[\times]c, d[$ an open rectangle in $\mathbb{R}^2$ with $a, b, c, d \in \mathbb{R}$ and $a < b$ and $c < d$ and $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ a continuous function defined on Ω. Is there any sign type condition on the boundary $\partial\Omega = \{a\} \times [c, d] \cup \{b\} \times [c, d] \cup [a, b] \times \{c\} \cup [a, b] \times \{d\}$ and on the components f_1 and f_2 such that the system of equations $f(x) = 0$ has a solution in Ω? Another simple generalization of Bolzano's theorem is to consider $n = 2$, $\Omega = B_{\mathbb{R}^2}(0, 1)$ the unit open euclidean ball in $\mathbb{R}^2$ and $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ a continuous function defined on Ω. Is there any sign type condition on the boundary $\partial\Omega = \mathbb{S}_2(1)$ and on the components f_1 and f_2 such that the system of equations $f(x) = 0$ has a solution in Ω? More generally, if $n \in \mathbb{N}$ is an arbitrary natural number, $\Omega \subseteq \mathbb{R}^n$ is a given general subset of $\mathbb{R}^n$ and $f = (f_1, ..., f_n)$ is a real-valued continuous function defined on Ω, is it possible to prove that the system of n variables and n equations $f(x) = 0$ has, at least, one solution in Ω? Is there a theory or concept that unifies all these questions from above? Yes, there is: the *Brouwer degree*.

1.2 The Brouwer degree and systems of equations

The aim of this section is to introduce mathematically the Brouwer degree. It looks hard to attribute concretely to someone the autorship of the existence of the Brouwer degree (see Theorem 2). Actually, it might be said that the Brouwer degree is a typical sample of stepped mathematical construction where the partial developments, begun around 1910, performed a really fundamental role in the final and general formulation (see [7]). Today it is known that there exists more than one way to introduce and prove the existence of the Brouwer degree but all of them are equivalent because of the unicity of the Brouwer degree (again see theorem 2) so, to some extent, it does not matter how to introduce or construct the degree. It is noteworthy to mention that the uniqueness of the Brouwer degree was not proved until the year 1973 by the mathematicians H. Amann and S. Weiss (see [13]).

The Brouwer degree is a powerful tool that allows to generalizate the Bolzano's theorem for continuous functions of several variables and several components. Clearly, the hypothesis $f(a)f(b) < 0$ in Bolzano's theorem means that $f(a)$ and $f(b)$ have different sign, so one can state the Bolzano's theorem in this new way.

Bolzano's theorem (revisited) Let $a, b \in \mathbb{R}$ be two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ a real-valued continuous function defined on the real interval $[a, b]$ such that $f(a)f(b) \neq 0$. If $\frac{1}{2}[sgn(f(b)) - sgn(f(a))] \neq 0$, then there exists $c \in]a, b[$ with $f(c) = 0$ or, equivalently, the equation $f(x) = 0$ has a solution in $]a, b[$.

Note that $sgn(f(a))$ and $sgn(f(b))$ make sense by virtue of the fact that $f(a)$ and $f(b)$ are not zero. The scalar $1/2$ multiplying the difference $sgn(f(b)) - sgn(f(a))$ is written just by aesthetic. The main goal is to replace the expression $\frac{1}{2}[sgn(f(b)) - sgn(f(a))]$ with another mathematical expression that is still valid on higher dimensions.

Exercise 6. *If $a, b \in \mathbb{R}$ are two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ a real-valued and continuous function of class $\mathcal{C}^1$ on $]a, b[$ such that $f(a)f(b) \neq 0$ and zero is a regular value of f ($x \in]a, b[, f(x) = 0 \Rightarrow f'(x) \neq 0$), prove that the set $f^{-1}(\{0\})$ is finite and*

$$\frac{1}{2}[sgn(f(b)) - sgn(f(a))] = \sum_{x \in f^{-1}(0)} sgn f'(x)$$

In the previous exercise, the summatory is considered zero when $f^{-1}(0) = \emptyset$. The advantage of the right-hand side expression is that it can be extended to greater dimensions.

Given $a, b \in \mathbb{R}$ two real numbers with $a < b$ and $f : [a, b] \to \mathbb{R}$ a continuous function defined on $[a, b]$ with $f(a)f(b) \neq 0$, the *Brouwer degree* $\deg_B(f,]a, b[, 0)$ of f at 0 relative to $]a, b[$ is

$$\deg_B(f,]a, b[, 0) = \frac{1}{2}\left(\frac{f(b)}{|f(b)|} - \frac{f(a)}{|f(a)|}\right) = \begin{cases} 1 & \text{si } f(a) < 0 < f(b) \\ -1 & \text{si } f(a) > 0 > f(b) \\ 0 & \text{si } f(a)f(b) > 0 \end{cases}$$

Example 3 (Brouwer degree of a polynomial)**.** *Let $n \in \mathbb{N}$ be a natural number, $a_n, ..., a_0 \in \mathbb{R}$ $n + 1$ real numebers with $a_n \neq 0$, $\mathrm{p}(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ a polynomial of order n with coefficients $a_n, ..., a_0$ and $\rho > 0$ big enough, then*

$$\deg_B(\mathrm{p},]-\rho, \rho[, 0) = \begin{cases} 0 & \textit{si n es par} \\ sgn(a_n) & \textit{si n es impar} \end{cases}$$

Henceforth, let $n \in \mathbb{N}$ be a natural number. Formally, the Brouwer degree in n dimensions is constructed in three steps.

The first step defines the Brouwer degree for a function f of class $\mathcal{C}^1$ in a non-empty, open and bounded subset Ω of $\mathbb{R}^n$ and at a regular value $y \in \mathbb{R}^n$ of f with $y \notin f(\partial\Omega)$.

In [5] and [12], it is proved that if $\Omega \subseteq \mathbb{R}^n$ is a non-empty, bounded and open subset of $\mathbb{R}^n$, $f : \overline{\Omega} \to \mathbb{R}^n$ is a continuous function on $\overline{\Omega}$ and of class $\mathcal{C}^1$ on Ω and $y \in \mathbb{R}^n$ is a regular point of f, then $f^{-1}(\{y\})$ is a finite set and the Brouwer degree of f at y relative to Ω is defined by the following integer number

$$\deg_B(f, \Omega, y) = \sum_{x \in f^{-1}(\{y\}) \cap \Omega} \operatorname{sgn} \det f'(x)$$

The second step defines the Brouwer degree for a function f of class $\mathcal{C}^1$ in a non-empty, open and bounded subset Ω of $\mathbb{R}^n$ but at an arbitrary point $y \in \mathbb{R}^n$ with $y \notin f(\partial\Omega)$. The mathematical tool that allows to remove the condition of regular value of f is the *Sard's lemma*. The Sard's lemma claims that the set of singular values of a $\mathcal{C}^1$ function defined on a non-empty and open subset Ω of $\mathbb{R}^n$ has Lebesgue measure zero, and thus define

$$\deg_B(f, \Omega, y) = \lim_{n \to +\infty} \deg_B(f, \Omega, y_n)$$

where $\{y_n\}$ is a sequence of regular values of f converging to y. In [5] and [12], it is shown that this is well-defined, namely it does not depend on the chosen sequence $\{y_n\}$.

Finally, the third and last step defines the Brouwer degree for a continuous function f on the closure of a non-empty, open and bounded subset Ω of $\mathbb{R}^n$ and at an arbitrary point $y \in \mathbb{R}^n$ with $y \notin f(\partial\Omega)$. The mathematical tool that allows to remove the condition of $\mathcal{C}^1$ function f is the *Weierstrass's approximation theorem*. The Weierstrass's approximation theorem claims that the for every continuous function $f \in \mathcal{C}(\overline{\Omega})$ there exists a sequence $\{f_n\} \subset \mathcal{C}^1(\overline{\Omega})$ such that $\{f_n\} \to f$ for the uniform norm $\| \cdot \|_0$ on $\mathcal{C}^0(\overline{\Omega})$, and thus define

$$\deg_B(f, \Omega, y) = \lim_{n \to +\infty} \deg_B(f_n, \Omega, y).$$

In [5] and [12], it is shown that this is well-defined, namely it does not depend on the chosen sequence $\{f_n\}$.

For the following, fix $n \in \mathbb{N}$ and let Σ be the set of all triples lists (f, Ω, y) where Ω is a non-empty, bounded and open subset of $\mathbb{R}^n$, $f : \overline{\Omega} \to \mathbb{R}^n$ is a continuous function defined on $\overline{\Omega}$ and $y \in \mathbb{R}^n$ a point of $\mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. Given $(f, \Omega, y) \in \Sigma$, an homotopy $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ is *admissible* if $H(x,t) \neq y$ for all $x \in \partial\Omega$ and $t \in [0,1]$. The reader can see the proof of the following theorem in [6] or [12], among others.

Theorem 2 (Existance and unicity of the Brouwer degree)**.** *There exists a unique application* $\deg : \Sigma \to \mathbb{Z}$ *(Brouwer degree), where* $\mathbb{Z}$ *stands for the set of integer numbers, such that*

(A1) ***Normalization:*** *if* $y \in \Omega$*, then* $\deg_B(Id, \Omega, y) = 1$*, where* Id *denotes the identity map.*

(A2) ***Additivity:*** *if* $\Omega_1, \Omega_2 \subseteq \Omega$ *are two open and disjoint subsets of* Ω *with* $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$*, then* $\deg_B(f, \Omega, y) = \deg_B(f, \Omega_1, y) + \deg_B(f, \Omega_2, y)$*.*

(A3) ***Homotopy invariance:*** *if* $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ *is a continuous and admissible homotopy for all* $t \in [0,1]$ *and* $y = y(t)$ *is a curve such that* $H(x,t) \neq y(t),\ x \in \partial\Omega,\ t \in [0,1]$, *then* $\deg_B(H(\cdot,t),\Omega,y(t))$ *is independent of* t.

Remark 2. *The axiom* $(A3)$ *is equivalent to the following two alternative axioms:*

- *if* $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ *is a continuous and admissible homotopy, then* $\deg_B(H(\cdot,t),\Omega,y)$ *is independent of* t.
- $\deg_B(f,\Omega,y) = \deg_B(f-y,\Omega,0)$.

There are massive amount of properties of the Brouwer degree. Here, the most important are presented.

As a direct consequence of the axiom $(A2)$ of the Brouwer degree, it follows these two results.

Proposition 1 (Excision). *If* $(f,\Omega,y) \in \Sigma$ *and* $\Lambda \subseteq \Omega$ *is a closed subset contained in* Ω *and* $y \notin f(\Lambda)$, *then*

$$\deg_B(f,\Omega,y) = \deg_B(f,\Omega \setminus \Lambda,y)$$

Proposition 2 (Additivity). *If* $m \in \mathbb{N}$, $(f,\Omega,y) \in \Sigma$ *and* $\Omega_1,...,\Omega_m \subseteq \Omega$ *open and disjoint subsets of* Ω *such that* $y \notin f(\overline{\Omega} \setminus \cup_{k=1}^m \Omega_k)$, *then* $(f,\Omega_k,y) \in \Sigma$ *for every* $k = 1,...,m$ *and*

$$\deg_B(f,\Omega,y) = \sum_{k=1}^{m} \deg_B(f,\Omega_k,y)$$

The next property is the required result that generalizes the Bolzano's theorem. Its proof is left as an exercise and it follows from the axiom $(A2)$ of the Brouwer degree.

Exercise 7 (Existance property). *Prove that if* $(f,\Omega,y) \in \Sigma$ *and* $\deg_B(f,\Omega,y) \neq 0$, *then the equation* $f(x) = y$ *has, at least, one solution in* Ω.

Proposition 3 (Dependence on the connected component). *If* $(f,\Omega,y) \in \Sigma$ *and* y_* *and* y *belong to the same connected component of* $\mathbb{R}^n \setminus f(\partial\Omega)$, *then* $\deg_B(f,\Omega,y) = \deg_B(f,\Omega,y_*)$

Proposition 4 (Dependence on the boundary). *If* $(f,\Omega,y),(g,\Omega,y) \in \Sigma$ *and* $f|_{\partial\Omega} = g|_{\partial\Omega}$, *then* $\deg_B(f,\Omega,y) = \deg_B(g,\Omega,y)$.

Proof. Define the continuous homotopy $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ by $H(x,t) = (1-t)f(x) + tg(x)$ for all $(x,t) \in \overline{\Omega} \times [0,1]$. For every $(x,t) \in \partial\Omega \times [0,1]$, clearly $H(x,t) = f(x) = g(x) \neq y$, hence the homotopy invariance of the Brouwer degree yields $\deg_B(f,\Omega,y) = \deg_B(g,\Omega,y)$. □

Exercise 8. *If* $a,b \in \mathbb{R}$ *are two real numbers with* $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *is a continuos function with* $f(a)f(b) \neq 0$, *prove that*

$$\deg_B(f,]a,b[,0) = \frac{1}{2}[sgnf(b) - sgnf(a)]$$

▶ *Hint*: use the proposition 4.

The Brouwer degree at zero for the particular case of linear functions is quite easy to calculate. Let $A \in \mathfrak{M}_n(\mathbb{R})$ be a regular matrix of order n, $\Omega \subseteq \mathbb{R}^n$ a non-empty, open and bounded subset of $\mathbb{R}^n$ containing the origin and consider the linear function $\varphi_A : \overline{\Omega} \to \mathbb{R}^n$ defined by $\varphi_A(x) = Ax$ for all $x \in \mathbb{R}^n$. On the one hand, since A is regular the only solution to the system $Ax = 0$ is $x = 0$, so $\varphi_A^{-1}(0) = 0$. On the other hand, since φ_A is linear it follows $\varphi_A' = A$. Consequently, $(\varphi_A, \Omega, 0) \in \Sigma$ and

$$\deg_B(\varphi_A, \Omega, 0) = \sum_{x \in \varphi_A^{-1}(0) \cap \Omega} \operatorname{sgn} \det \varphi_A'(x) = \operatorname{sgn} \det A$$

For example, if $\Omega \subseteq \mathbb{R}^n$ a non-empty, open and bounded subset of $\mathbb{R}^n$ containing the origin, one has $\deg_B(-Id, \Omega, 0) = (-1)^n$. More properties of Brouwer degree can be found in [5] and [12].

Example 4. *Let $\rho > 0$ be a positive number, $B = B_{\mathbb{R}^2}(0, \rho)$ the open ball in $\mathbb{R}^2$ centered at the origin of radius ρ and $f : \overline{B} \to \mathbb{R}^2$ the linear function given by*

$$f(x, y) = (2x + y, x - 2y) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall (x, y) \in \overline{B}$$

The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ has determinant equal to -5, then, from the previous paragraph, $\deg(f, B, 0) = sgn(-5) = -1$. *At this point, the existence property provide the existence of one solution to the system of equations*

$$\begin{cases} 2x + y = 0 \\ x - 2y = 0 \end{cases}$$

The example 4 is not interesting at all. Everyone knows that the previous system of equations have one, and only one, solution $(x, y) = (0, 0)$ and it is not necessary to apply the Brouwer degree theory here, but what about the following system of equations? Can the reader stop and think for a moment how to obtain the existance of solution for this second system of equations?

$$(*) \quad \begin{cases} 2x + y + \sin(x + y) = 0 \\ x - 2y + \cos(x + y) = 0 \end{cases}$$

This system admit solution and it belongs to $B(0, r)$ provided that $r > 1/\sqrt{5}$ and the way to prove it is using the Brouwer degree. First, suppose that the system has a solution (x_0, y_0), then

$$\begin{cases} 2x_0 + y_0 = -\sin(x_0 + y_0) \\ x_0 - 2y_0 = -\cos(x_0 + y_0) \end{cases} \Rightarrow \begin{cases} (2x_0 + y_0)^2 = (-\sin(x_0 + y_0))^2 \\ (x_0 - 2y_0)^2 = (-\cos(x_0 + y_0))^2 \end{cases}$$

Adding these two equations,

$$(2x_0 + y_0)^2 + (x_0 - 2y_0)^2 = 1$$

and simplifying, it follows

$$x_0^2 + y_0^2 = 1/5.$$

Take $r > 1/\sqrt{5}$, $\Omega = B(0, r)$ and consider the funtions $f \in \mathcal{C}(\overline{\Omega})$ and $g \in \mathcal{C}(\overline{\Omega})$ given by

$$f(x, y) = (2x + y, x - 2y) \quad \forall (x, y) \in \overline{\Omega}$$

$$g(x, y) = (\sin(x + y), \cos(x + y)) \quad \forall (x, y) \in \overline{\Omega}$$

and the continuous homotopy $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^2$ defined by

$$H(x, y, t) = f(x, y) + tg(x, y) \quad \forall ((x, y), t) \in \overline{\Omega} \times [0, 1]$$

If $(x, y) \in \partial\Omega$ and $\lambda \in [0, 1]$, then

$$\|H(x, y, t)\|^2 \geq \|f(x, y)\|^2 - t\|g(x, y)\|^2 \geq \sqrt{5} - \lambda > 0$$

where it has been used that $\|g(x, y)\|^2 = 1$ and $\|f(x, y)\|^2 = 5(x^2 + y^2) = 5r > 5/\sqrt{5} = \sqrt{5}$. This implies that H is an admissible homotopy and, in view of the homotopy invariance of the Brouwer degree, $\deg_B(f + g, \Omega, 0) = \deg_B(f, \Omega, 0)$, but f is just the linear function of the example 4 and $0 \in \Omega$, so

$$\deg_B(f + g, \Omega, 0) = \deg_B(f, \Omega, 0) = -1 \neq 0$$

Finally, the existence property concludes the required existence of solution for the system of equations $(*)$. Roughly speaking, note that basically the priori bound of solutions leads to existance of solution and this explains one famous quote of the mathematician J. Mawhin: *I am limited, therefore I exist!*.

Exercise 9. *Prove the existence of solution of*

$$\begin{cases} 8x + 6y + \frac{\log(x^2+2)}{x^2+y^2+1} - \sin(y - 7) + 15 = 0 \\ 3x - y + \exp(-y^2 - 4) + \cos(xy) + 1 = 0 \end{cases}$$

1.3 Brouwer fixed point theorem

A **fixed point** of a function $f : \Omega \to \mathbb{R}^n$ defined on a non-empty subset Ω of $\mathbb{R}^n$ is a point $x \in \Omega$ such that $f(x) = x$. This section is focused on the *Brouwer fixed point theorem*, a result that provides the existence of a fixed point for continuous functions defined on a subset homeomorphic to a non-empty, convex, closed and bounded subset of $\mathbb{R}^n$, in its most general version. Let us start with the most simple and classical version:

Theorem 3 (Brouwer's fixed point)**.** *Let $n \in \mathbb{N}$ be a natural number, $\rho \in \mathbb{R}^+$ a positive real number, $\overline{B} = \overline{B}_{\mathbb{R}^n}(0, \rho)$ the closed ball in $\mathbb{R}^n$ centered at the origin of radius ρ and $f : \overline{B} \to \overline{B}$ a continuous function defined on $\overline{B}$ with values on $\overline{B}$. Then f has, at least, one fixed point in $\overline{B}$. Equivalently, the equation $f(x) = x$ has solution in Ω.*

Proof. Define the function $g : \overline{B} \to \mathbb{R}^n$ by $g(x) = x - f(x)$ for all $x \in \overline{B}$ and the continuous homotopy $H : \overline{B} \times [0,1] \to \mathbb{R}^n$ by $H(x,t) = x - tf(x)$ for all $(x,t) \in \overline{B} \times [0,1]$. If there exists a point $x \in \partial B$ such that $f(x) = x$, f has a fixed point on $\partial B_{\mathbb{R}^n}(0,\rho)$ and the proof is finished. If not, on the one hand, if $x \in \partial B$ and $t \in [0,1[$, $x - tf(x) = 0$ implies $\rho = \|x\| = t\|f(x)\| < \|f(x)\| \leq \rho$ (contradiction) and, on the other hand, if $x \in \partial B$ and $t = 1$, $x - tf(x) = 0$ implies $\rho = \|x\| = \|f(x)\|$ but the assumption is that $f(x) \neq x$ for every $x \in \partial B_{\mathbb{R}^n}(0,\rho)$ (contradiction). The homotopy invariance, together with the normalization, of the Brouwer degree yield $\deg_B(Id - f, B_{\mathbb{R}^n}(0,\rho), 0) = \deg_B(Id, B_{\mathbb{R}^n}(0,\rho), 0) = 1 \neq 0$. Finally, the existance property of the Brouwer degree provides a point $x_* \in B_{\mathbb{R}^n}(0,\rho)$ such that $(Id - f)(x_*) = 0$ or, equivalently, $f(x_*) = x_*$ and f has a fixed point on $B_{\mathbb{R}^n}(0,\rho)$. □

The Brouwer's fixed point theorem was first proved in dimension three by Brouwer himself in 1909. Next year, in 1910, Hadamard extended the result to an arbitrary dimension and finally in 1912, Brouwer also proved it for every dimesion but now using the so called Brouwer degree.

Remark 3. *In dimension one, Brouwer's fixed point theorem is equivalent to Bolzano's theorem.*

Theorem 4. *Let $n \in \mathbb{N}$ be a natural number, $K \subseteq \mathbb{R}^n$ a non-empty, convex, closed and bounded subset of $\mathbb{R}^n$ and $f : K \to K$ a continuous function defined on K and with values on K, then f has, at least, one fixed point in K.*

Note that the proof of the previous theorem does not work here because the interior of K can be the empty set. In $\mathbb{R}$, there are not non-empty, convex, closed and bounded subsets with empty interior owing to the fact that convex subsets of $\mathbb{R}$ are just the real intervals. In greater dimensions, this already happens: think in a non-empty and closed segment of $\mathbb{R}^2$, for example. The proof of the previous theorem is based on the *Dugundji extension's theorem* that is next stated. The reader can see the proof of it in [15].

Theorem 5 (Dugundji)**.** *Let $K \subset \mathbb{R}^n$ a non-empty and compact subset of $\mathbb{R}^n$, $\Lambda \subset \mathbb{R}^n$ a non-empty subset of $\mathbb{R}^n$ and $f : K \to \Lambda$ a continuous function defined on K. Then there exists a function $F : \mathbb{R}^n \to \mathbb{R}^n$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that*

(a) F is continuous

(b) $F|_K = f$

(c) $F(\mathbb{R}^n) \subseteq \overline{conv(\Lambda)}$

Proof of theorem 4. In the case of the theorem 4, $\Lambda = K$ and K is a non-empty, convex and compact (closed and bounded) subset of $\mathbb{R}^n$, so the Dugundji theorem asserts the existence of a continuous extension function $F : \mathbb{R}^n \to \mathbb{R}^n$ of f with $F(\mathbb{R}^n) \subseteq \overline{\text{conv}(K)}$. Since K is convex and closed, $\overline{\text{conv}(K)} = K$ and therefore the image of F is contained in K and so the restriction to K of F, i.e. $f = F|_K : K \to K$. Finally, since K is bounded, one can take $R > 0$ such that $K \subset \overline{B}(0,R)$ and consider the restriction $F|_{\overline{B}(0,R)} : \overline{B}(0,R) \to K \subset \overline{B}(0,R)$, then the theorem 3 provides a point $x_* \in \overline{B}(0,R)$ with $F|_{\overline{B}(0,R)}(x_*) = x_*$ but $F(\overline{B}(0,R)) \subset K$, so $x_* \in K$ and then by the fact that $F|_K = f$, it is concluded that $f(x_*) = x_*$ or, equivalently, x_* is a fixed point of f in K. □

To conclude, the proof of the most general statement is left as an easy exercise.

Theorem 6. *Let $n \in \mathbb{N}$ be a natural number, $\Omega \subset \mathbb{R}^n$ a subset of $\mathbb{R}^n$ that is homeomorphic to a non-empty, convex, closed and bounded subset K of $\mathbb{R}^n$ and $f : \Omega \to \Omega$ a continuous function defined on Ω with values on Ω. Then f has, at least, one fixed point in Ω.*

Exercise 10. *Prove the previous theorem.*

More applications of Brouwer degree can be found in [5] and [12].

A really hard and interesting question concerning fixed points of continuous functions is now also left as an exercise for the reader. In dimension one, it is effortless to imagine a real-valued and continuous function defined on a non-trivial, closed interval with only one fixed point, or with only two fixed points, or with only three fixed points,... or with infinite fixed points too! (and not necessarily the identity map). The following exercise establishes that there always exists a continuous function with exactly the fixed points given by a closed set.

Exercise 11. *Prove that if $\rho \in \mathbb{R}^+$ is a positive real number and $C \subset \overline{B}_{\mathbb{R}^n}(0, \rho)$ is a non-empty and closed subset contained in the closed ball centered at the origin and of radius ρ, then there exists a continuous function $f : \overline{B}_{\mathbb{R}^n}(0, \rho) \to \overline{B}_{\mathbb{R}^n}(0, \rho)$ such that the set of the fixed points of f is exactly the set C.*

▶ *Hint*: use the distance function $d(\cdot, C)$.

1.4 Introduction and motivation for the Leray-Schauder degree

What happend when one walks into infinte-dimensional spaces? There is a big problem: it is not possible to define a *topological degree* similar to Brouwer degree when the considered space is infinite-dimensional. First of all, some concepts and properties of the infinite dimension are recalled next.

A basis $\mathcal{B}$ of a real finite-dimensional vector space X is a set of elements of X that has both the linear independence and the spanning property.

$$\{e_1, ..., e_n\} \subset X \text{ linearly independant} \iff \forall \lambda_1, ..., \lambda_n \in \mathbb{R}, \sum_{k=1}^{n} \lambda_k e_k = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$$

$$\{e_1, ..., e_n\} \subset X \text{ spanning set of } X \iff \forall x \in X,\ \exists \lambda_1, ..., \lambda_n \in \mathbb{R} : x = \sum_{k=1}^{n} \lambda_k e_k$$

The coefficients of this linear combination are referred to as components or coordinates on $\mathcal{B}$ of the vector. A real vector space X has finite dimension $n \in \mathbb{N}$ if there exists a basis of $\mathcal{B} \subset X$ with exactly n vectors. Every vector space admits basis and, by the basis theorem, if there is one basis of X formed by n components then all the bases of X have also n components. Equivalently, $\mathcal{B}$ is a basis of X if, and only if, every element of X may be written in a unique way as a (finite) linear combination of elements of $\mathcal{B}$:

$$\{e_1, ..., e_n\} \subset X \text{ base de } X \iff \forall x \in X,\ \exists! \lambda_1, ..., \lambda_n \in \mathbb{R} : x = \sum_{k=1}^{n} \lambda_k e_k$$

A real vector space X has infinite dimension if it has not finite dimension. For instance, if $a, b \in \mathbb{R}$ are two real numbers with $a < b$, then the space $\mathcal{C}([a,b],\mathbb{R})$ of continuous and real-valued funtions defined on $[a,b]$ has infinite dimension. The easiest way to prove this is to find a linearly independant and infinite subset of $\mathcal{C}([a,b],\mathbb{R})$. The set $\mathrm{P} = \{x^n : n \in \mathbb{N} \cup \{0\}\}$ is contained in $\mathcal{C}([a,b],\mathbb{R})$, has infinite elements and is linearly independant. Unfortunately, it is not a spanning set, so P is not a basis of $\mathcal{C}([a,b],\mathbb{R})$.

Exercise 12. *Prove that $C = \{x^n : n \in \mathbb{N} \cup \{0\}\}$ is linearly independent but is not a basis of $\mathcal{C}([a,b],\mathbb{R})$.*

In fact, every linearly independent subset $\mathcal{B}$ of a vector space X can be extended into a basis of X, so it is clear that there exists a basis of $\mathcal{C}([a,b],\mathbb{R})$ that contains P. In general, it turns imposible to find explicitly a basis of a infinite-dimensional space, like $\mathcal{C}([a,b],\mathbb{R})$. Moreover, as a consequence of Baire's theorem, the bases of $\mathcal{C}([a,b],\mathbb{R})$ are uncountable infinite sets.

A **norm** on a real vector space X is a map $\|\cdot\| : X \to \mathbb{R}$ with the following properties

- $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if, and only if, $x = 0$.
- $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in X$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A normed space $(X, \|\cdot\|)$ is a vector space X equipped with a norm $\|\cdot\|$.

One can equipped $\mathbb{R}^n$ with so many different norms, for example

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p} \quad \forall x = (x_1, ..., x_n) \in \mathbb{R}^n,\ \forall p \in [1, \infty[$$

$$\|x\|_\infty = \max_{1 \leq k \leq n} \{|x_k|\} \quad \forall x = (x_1, ..., x_n) \in \mathbb{R}^n$$

One can equipped $\mathcal{C}([a,b],\mathbb{R})$ with so many different norms, for example

$$\|f\| = \int_a^b |f(t)|\, dt \quad \forall f \in \mathcal{C}([a,b],\mathbb{R})$$

$$\|f\|_0 = \max_{a \leq t \leq b} \{|f(t)|\} \quad \forall f \in \mathcal{C}([a,b],\mathbb{R})$$

It is known that there are great differences between the finite and the infinite dimensions. All norms in $\mathbb{R}^n$ are equivalent, while this is not true for $\mathcal{C}([a,b],\mathbb{R})$. The equivalence of norms is significant, because it assures that one can study the topological properties independent of the considered norm. Another remarkable difference between the finite and the infinite dimensions is the Bolzano-Weierstrass theorem, which holds only for finite dimension. The Bolzano-Weierstrass theorem states that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. This is false for infinite dimension as the reader can see in the next exercise.

Exercise 13. *If $X = (\mathcal{C}([a,b],\mathbb{R}), \|\cdot\|_0)$, the sequence $\{f_n\}$ of X given by $f_n(t) = t^n$ for all $t \in [a,b]$ and $n \in \mathbb{N}$ is bounded and it has not a convergent subsequence.*

This section started by confirming the imposibility of an analogous of Brouwer degree in infinite-dimensional spaces. The reason why this happends lies on the Brouwer fixed point theorem. Imagine for a moment that one has already construct a topological degree in infinite dimnesion and similar to Brouwer degree, then the Brouwer degree theorem would still remains true since only the axioms $(A1)-(A2)-(A3)$ are used in its proof. Here is the problem! the Brouwer fixed point is not true for infinite-dimensional spaces, in other words, for each infinite-dimensional space there is a continuous function on the closed unit ball into itself that has no fixed points. This statement is really hard to prove, but in order to clarify it, it is showed a particular example now:

Consider the space $X = (\mathbf{c}_0, \|\cdot\|)$ of all the convergent sequences of real numbers with limit zero and the norm given by $\|\{x_n\}\| = \max_{n\in\mathbb{N}} |x_n|$ for all $\{x_n\} \in \mathbf{c}_0$. Clearly, X is a infinte-dimensional space as $\{e_n : n \in \mathbb{N}\}$ is a infinite and linearly independant subset of X, where $e_n = \{0,...,0,\overset{n)}{1},0,...0\}$ for each $n \in \mathbb{N}$. Note that $\{e_n\}$ is not a basis of X: the sequence $\{1/n\} \in X$ cannot be expressed by a finite linear combination of elements of $\{e_n : n \in \mathbb{N}\}$. Futhermore, it is proved with no difficulty that X is a Banach space. Here, the closed unit ball is

$$\overline{B}_X(0,1) = \{\{x_n\} \in \mathbf{c}_0 : \|\{x_n\}\| \leq 1\} = \{\{x_n\} \in \mathbf{c}_0 : |x_n| \leq 1, n \in \mathbb{N}\}$$

The function $f : \overline{B}_X(0,1) \to \overline{B}_X(0,1)$ that maps every sequence $\{x_n\}$ to the sequence

$$f(\{x_n\}) = \left\{\frac{1 + \| \{x_n\} \|}{2}, x_1, x_2, ..., x_n, ...\right\}$$

is well-defined, continuous but it has no fixed points. Indeed, if $\{x_n\} \in \mathbf{c}_0$ and $\|x_n\| \leq 1$, then $\|f(\{x_n\})\| = \max\left\{\frac{1+\|\{x_n\}\|}{2}, |x_1|, |x_2|, ..., |x_n|, ...\right\} \leq 1$ and, in addition, is more than continuous, f is 1-lipschitz: given $\{x_n\}, \{y_n\} \in \mathbf{c}_0$,

$$\begin{aligned}
\|f(\{x_n\}) - f(\{y_n\})\| &= \max_{n\in\mathbb{N}} |f(\{x_n\}) - f(\{y_n\})| = \\
&= \max_{n\in\mathbb{N}} \left\{\frac{|\|\{x_n\}\| - \|\{y_n\}\||}{2}, |x_1 - y_1|, |x_2 - y_2|, ...\right\} \leq \\
&\leq \|\{x_n\} - \{y_n\}\|
\end{aligned}$$

Finally, is clear from $f(\{x_n\}) = \{x_n\}$ that $\{x_n\}$ must be the constant sequence

$$x_n = \frac{1 + \|\{x_n\}\|}{2}, \quad n \in \mathbb{N}$$

which does not converge to zero and this complete the reasoning, there is no sequence $\{x_n\}$ in X such that $f(\{x_n\}) = \{x_n\}$.

All this does not mean that it is imposible to define a useful degree in infinite-dimensional spaces. In fact, mathematicians from 1930s kept trying to define this topological tool in infinite-dimensional spaces because of its diverse applications to integral equations or boundary-value problems. Mathematicians realized that the functions that appears in the applications satisfy an additional property to due continuity and this property helped them to construct a "Brouwer fixed point theorem" version for infinite dimension. Concretely, this was done by the French and German mathematicians, J. Leray and J. Schauder, respectively. They both create a degree for functions of the form $I - T$ where I is the identity map and T is a *compact* function.

Definition 1. *Given $(X, \|\cdot\|)$ a (real) normed space and $\Omega \subseteq X$ a non-empty, bonded and open subset of X, an operator $T : \overline{\Omega} \to X$ is* ***compact*** *if*

1. *T is continuous.*
2. *$T(\overline{\Omega})$ is relatively compact ($\Leftrightarrow \overline{T(\overline{\Omega})}$ is compact).*

Remark 4. *In the situation of the previous definition, note that if $\dim X < \infty$, then T is compact if, and only if, T is continuous.*

Remark 5. *If $\dim X = \infty$, there are continuous functions that are not compact. The identity map is continuous but not compact, otherwise $Id(\overline{B}_X(0,1)) = \overline{B}_X(0,1)$ would be relatively compact and also compact. However, the closed unit ball in infinite-dimensional spaces is never compact!. Actually, $\dim X < \infty$ if, and only if, $\overline{B}_x(0,1)$ is compact.*

Remark 6. *If $X = (\mathbb{R}^n, \|\cdot\|)$ for some $n \in \mathbb{N}$, then $\Omega \subseteq X$ is relatively compact if, and only if, Ω is bounded. Of course, it does not matter what norm is being considered due to the fact that all the norms on $\mathbb{R}^n$ are equivalent.*

Remark 7 (Arzelà-Ascoli theorem)**.** *If $X = (\mathcal{C}^0([a,b], \mathbb{R}), \|\cdot\|_0)$ for some $a, b \in \mathbb{R}$ with $a < b$, then $\Omega \subseteq X$ is relatively compact if, and only if, Ω is*

1. *uniformly bounded: $\exists M > 0 : |f(x)| \leq M,\ \forall x \in [a,b],\ \forall f \in X$*
2. *equicontinuous: $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 : x, y \in [a,b], |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon,\ \forall f \in \Omega$*

For a fixed but arbitrary function f, the uniformly boundness assures that f is bounded, but this is already known because f is continuous and defined on the compact $[a,b]$.
For a fixed but arbitrary function f, the equicontinuity assures that f is uniformly continuous, but this is already known by the Heine theorem.

A really useful sufficient condition for equicontinuity follows from the mean value theorem and it is stated next.

Sufficient condition for equicontinuity. If the functions of $\Omega \subset \mathcal{C}^0([a,b],\mathbb{R})$ are derivable and there exists $K \geq 0$ such that $|f'(x)| \leq K$ for all $x \in [a,b]$ and for all $f \in \Omega$, then Ω is equicontinuous.

Compact operators verify the following property that allowed to define a degree in infinite-dimensional spaces whose proof can be read in [12].

Theorem 7. *Let $(X, \|\cdot\|)$ be a Banach space, $\Omega \subseteq X$ a non-empty, bonded and open subset of X and $T : \overline{\Omega} \to X$ a compact function, then there exists a sequence $\{T_n\}$ of functions defined on $\overline{\Omega}$ and with values on X such that*

1. *T_n is compact for all $n \in \mathbb{N}$.*
2. *T_n converges uniformly to T in $\overline{\Omega}$.*
3. *for every $n \in \mathbb{N}$, $Im(T_n) \subset X_n$, where X_n is a subspace of finte-dimension of X.*

1.5 The Leray-Schauder degree

For the following, let Σ be the set of all triples lists $(I - T, \Omega, y)$ where Ω is a non-empty, bounded and open subset of X, I is the identity map on Ω, $T : \overline{\Omega} \to X$ is a compact function defined on $\overline{\Omega}$ and $y \in X$ a point of X such that $y \notin (I-T)(\partial\Omega)$. Given $(I-T, \Omega, y) \in \Sigma$, an homotopy $H : \overline{\Omega} \times [0,1] \to X$ is *admissible* if $I(x) - H(x,t) \neq y$ for all $x \in \partial\Omega$ and $t \in [0,1]$. The reader can see the proof of the following theorem in [6] or [12], among others.

Theorem 8 (Leray-Schauder, 1934). *Let $(X, \|\cdot\|)$ be a Banach space, then there exists a unique application $\deg_{LS} : \Sigma \to \mathbb{Z}$ (Leray-Schauder degree) such that*

(B1) ***Normalization:*** *if $y \in \Omega$, then $\deg_{LS}(I, \Omega, y) = 1$.*

(B2) ***Additivity:*** *if $\Omega_1, \Omega_2 \subseteq \Omega$ are two open and disjoint subsets of Ω with $y \notin (I-T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then $\deg_{LS}(f, \Omega, y) = \deg_{LS}(f, \Omega_1, y) + \deg_{LS}(f, \Omega_2, y)$.*

(B3) ***Homotopy invariance:*** *if $H : \overline{\Omega} \times [0,1] \to X$ is a compact and admissible homotopy for all $t \in [0,1]$ $(I(x) - H(x,t) \neq y(t),\ x \in \partial\Omega, t \in [0,1])$, then $\deg_{LS}(H(\cdot,t), \Omega, y(t))$ is independent of t.*

Remark 8. *When the dimension of the Banach space X is finite, the Leray-Schauder degree is nothing but the Brower degree!.*

The Leray-Schauder degree enjoys of similar properties of Brouwer degree.

Proposition 5 (Excision). *If $(I-T, \Omega, y) \in \Sigma$ and $\Lambda \subseteq \Omega$ is a closed subset contained in Ω and $y \notin f(\Lambda)$, then*

$$\deg_{LS}(I - T, \Omega, y) = \deg_{LS}(I - T, \Omega \setminus \Lambda, y)$$

Proposition 6 (Additivity). *If $m \in \mathbb{N}$, $(fI - T, \Omega, y) \in \Sigma$ and $\Omega_1, ..., \Omega_m \subseteq \Omega$ open and disjoint subsets of Ω such that $y \notin (I - T)(\overline{\Omega} \setminus \cup_{k=1}^{m} \Omega_k)$, then $(I - T, \Omega_k, y) \in \Sigma$ for every $k = 1, ..., m$ and*

$$\deg_{LS}(I - T, \Omega, y) = \sum_{k=1}^{m} \deg_{LS}(I - T, \Omega_k, y)$$

Proposition 7 (Existence property). *If $(I - T, \Omega, y) \in \Sigma$ and $\deg_{LS}(f, \Omega, y) \neq 0$, then the equation $x - Tx = y$ has, at least, one solution in Ω.*

Proposition 8 (Dependence on the connected component). *If $(I - T, \Omega, y) \in \Sigma$ and y_* and y belong to the same connected component of $X \setminus (I - T)(\partial\Omega)$, then $\deg_{LS}(f, \Omega, y) = \deg_{LS}(f, \Omega, y_*)$*

Proposition 9 (Dependence on the boundary). *If $(I - T, \Omega, y), (I_T^*, \Omega, y) \in \Sigma$ and $T|_{\partial\Omega} = T^*|_{\partial\Omega}$, then $\deg_{LS}(I_T, \Omega, y) = \deg_{LS}(I - T^*, \Omega, y)$.*

More properties of Leray-Schauder degree can be found in [6], [12] and [14].

1.6 The Schauder fixed point theorem and boundary value problems

Theorem 9 (Schauder, 1930). *Let X be a real Banach space and $K \subseteq X$ a non-empty, convex, bounded and closed subset of X and $T : K \to K$ a compact funcion defined on K with values on K, then T has, at least, one fixed point in K. Equivalently, the equation $Tx = x$ has solution in K.*

Exercise 14. *Prove the Schauder fixed point theorem.*

Exercise 15 (Schauder fixed point theorem extended). *Prove or disprove the following claim: if X is a real Banach space, $T : D \to D$ is a compact map and D is homeomorphic to a non-empty, convex, bounded and closed subset of X, then T has, at least, one fixed point in D.*

Exercise 16. *Prove that if X is a Banach space with $\dim X = \infty$ and $K \subset X$ is compact, then $K^\circ = \emptyset$.*

Corollary 3. *Let X be a real Banach space and $K \subseteq X$ a non-empty, convex and compact subset of X and $T : K \to K$ a continuous funcion defined on K with values on K, then T has, at least, one fixed point in K. Equivalently, the equation $Tx = x$ has solution.*

One of the most important applications of the Leray-Schauder degree lies on boundary values problems. For instance, given $a, b \in \mathbb{R}$ with $a < b$ and a continuous and bounded function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, the problem

$$(BVP) \begin{cases} -x''(t) = f(t, x(t)), \ t \in]a, b[\\ x(a) = x(b) = 0 \end{cases}$$

has, at least, one solution in $\mathcal{C}^2([a,b],\mathbb{R})$. This problem is translated to an integral equation via the associated Green's function $K \in \mathcal{C}([a,b]\times[a,b],\mathbb{R})$. It is proved that $x \in \mathcal{C}^2([a,b],\mathbb{R})$ is a solution of (BVP) if, and only if, $x \in \mathcal{C}([a,b],\mathbb{R})$ and

$$x(t) = \int_a^b K(t,s)f(s,x(s))\ ds \quad \forall t \in [a,b].$$

Exercise 17. *Consider the functional space $X = (\mathcal{C}([a,b],\mathbb{R}), \|\cdot\|_0)$ and the map $T : X \to X$ defined by*

$$T(x)(t) = \int_a^b K(t,s)f(s,x(s))\ ds \quad \forall t \in [a,b]$$

for every $x \in X$, where $K \in \mathcal{C}([a,b]\times[a,b],\mathbb{R})$. Prove the following assertions:

a) T is well-defined.

b) T is continuous.

c) If $B \subset X$ is bounded, then $T|_B : B \to X$ is compact (use Arzelà-Ascoli theorem).

d) Apply the Schauder fixed point to get a fixed point of T, meaning a solution of

$$x(t) = \int_a^b K(t,s)f(s,x(s))\ ds$$

Chapter 2

Sobolev spaces

The chapter is divided into four sections (Introduction, Lebesgue spaces, Sobolev spaces and Eigenvalues of linear Dirichlet boundary value problem). The main bibliography used here is [8].

2.1 Introduction

The variational method is an analitycal method used to solve problems concerning partial differential equations, among others. This method consists of defining a functional on a certain functional space so that its critical points turn out to be solutions of the original problem. In other words, we solve one partial differential equation suject to some conditions by finding critical points of a suitable funcional defined on a suitable functional space, which is often easier to manage. Let's think in the following example,

$$(P)\ \begin{cases} u'' = u + f, \ x \in]0, \pi[\\ u(0) = u(\pi) = 0 \end{cases}$$

where $u = u(x)$ is the unknown function and $f : \mathbb{R} \to \mathbb{R}$ is a given and continuous function. It seems logic to set the functional space $X = \{v \in \mathcal{C}^2[0, \pi] : v(0) = v(\pi) = 0\}$ where to find the possible solutions to (P). Imagine that $u \in \mathcal{C}^2[0, \pi]$ is a solution of (P) and $v \in X$ is an arbitrary function of X, then multiplying the equation $u'' = u + f$ by v and integrating on $[0, \pi]$, one gets

$$-\int_0^\pi u'(x)v'(x)\ dx = \int_0^\pi u(x)v(x)\ dx + \int_0^\pi f(x)v(x)\ dx \tag{2.1}$$

after applying the integration by parts formula and using the fact that $v(0) = v(\pi) = 0$,

$$\int_0^\pi u''(x)v(x)\ dx = u'(x)v(x)]_0^\pi - \int_0^\pi u'(x)v'(x)\ dx = -\int_0^\pi u'(x)v'(x)\ dx$$

Since $v \in X$ is arbitrary, the equation (2.1) is equivalent to write

$$\int_0^\pi [u'(x)v'(x) + u(x)v(x) + f(x)v(x)]\ dx = 0 \quad \forall v \in X \tag{2.2}$$

Conversely, if equation (2.2) holds true for all $v \in X$, then it must be $u'' - u - f = 0$ on $[0, \pi]$. This is clear from the fact that $u'' - u - f$ is continuous on $[0, \pi]$, so if there exists one point $x_0 \in [0, \pi]$ with $u''(x_0) - u(x_0) - f(x_0) > 0$ (analogous for < 0) it follows that there exists an open interval J centered at x_0 such that $u'' - u - f$ is non-negative on J. Finally, one can define a function $v_0 \in X$ with the property of being zero outside J and non-negative inside J and consequently it should be

$$\int_0^\pi [u''(x) - u(x) - f(x)]v_0(x)\; dx > 0$$

which is a contradiction.

The aim here is to define a functional $I : X \to \mathbb{R}$ such that the "derivative" $I'(u)$ (this will be defined rigorously later) at a function $u \in X$ is given by

$$I'(u)(v) = \int_0^\pi \left[u'(x)v'(x) + u(x)v(x) + f(x)v(x)\right]\; dx \quad \forall v \in X$$

so this means that $u \in X$ is a solution of (P) if, and only if, $I'(u) \equiv 0$. By definition, u is a *critical point* of a functional I when $I'(u) \equiv 0$. The functional I we are looking for is nothing but

$$I(u) = \frac{1}{2}\int_0^\pi (u'(x))^2\; dx + \frac{1}{2}\int_0^\pi u(x)^2\; dx + \int_0^\pi f(x)u(x)\; dx \quad \forall u \in X$$

One typical way to see that a certain function u is a critical point of I is just proving that u is a global minimum or maximum or a saddle point of I, actually.

Consider in X the following norm

$$\|u\| = \left(\int_0^\pi |u(x)|^2\; dx + \int_0^\pi |u'(x)|^2\; dx\right)^{1/2} \qquad \forall u \in X \tag{2.3}$$

The functional I is bounded from below. Indeed,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^\pi \left[(u(x)')^2 + u(x)^2)\right]\; dx + \int_0^\pi f(x)u(x)\; dx \geq \\ &\geq \frac{1}{2}\int_0^\pi \left[(u(x)')^2 + u(x)^2)\right]\; dx + \left(\int_0^\pi f(x)^2\; dx\right)^{1/2}\left(\int_0^\pi u(x)^2\; dx\right)^{1/2} \geq \\ &\geq \int_0^\pi (u'(x))^2\; dx - \frac{1}{2}\int_0^\pi f(x)^2\; dx \geq -\frac{1}{2}\int_0^\pi f(x)^2\; dx \end{aligned}$$

where we used the Hölder inequality firstly and the fact that $1/2t^2 - at \geq -a^2/2$ for all $t \in \mathbb{R}$ secondly. Besides, I is coercive, that is $I(u) \to +\infty$ as $\|u\| \to +\infty$. In order to prove this, one just write the following inequalities,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_0^\pi \left[(u(x)')^2 + u(x)^2)\right]\,dx + \int_0^\pi f(x)u(x)\,dx \geq \\
&\geq \frac{1}{2}\int_0^\pi \left[(u(x)')^2 + u(x)^2)\right]\,dx + \left(\int_0^\pi f(x)^2\,dx\right)^{1/2}\left(\int_0^\pi u(x)^2\,dx\right)^{1/2} \geq \\
&\geq \tfrac{1}{4}\int_0^\pi (u'(x))^2\,dx + \tfrac{1}{4}\int_0^\pi u(x)^2\,dx + \tfrac{1}{4}\int_0^\pi u(x)^2\,dx - \left(\int_0^\pi f(x)^2\,dx\right)^{1/2}\left(\int_0^\pi u(x)^2\,dx\right)^{1/2} \geq \\
&\geq \frac{1}{4}\|u\|^2 - cte
\end{aligned}$$

At this point, it is known that I is bounded from below, coercive and, although it was not proved, is $\mathcal{C}^1$ on X. It is easy to check the following result when the considered functional space has finite-dimension:

Proposition 10. *If $N \in \mathbb{N}$ and $I : \mathbb{R}^N \to \mathbb{R}$ is a $\mathcal{C}^1$-funtional on $\mathbb{R}^N$ that is bounded from below (resp. from above) and coercive, then I has global minimum (resp. global maximum).*

Notwithstanding, this is false when $\mathbb{R}^N$ is replaced by an arbitrary space X of arbitrary dimension. In order to be able to apply a similar result which provides that a functional $I : X \to \mathbb{R}$ attains its global minimum at some point in X, some more additional hypothesis are needed: X must be a Banach space. For instance, it is not hard to think in a $\mathcal{C}^1$-functional $I : \mathbb{Q} \to \mathbb{R}$ bounded from below and coercive, but without a global minimum in $\mathbb{Q}$. Note that $\mathbb{Q}$ is not complete, since the sequence $\{3, 3.1, 3.14, 3.141, 3.1415, ...\}$ is a Cauchy sequence of rational numbers and it does not converge to a rational number. To avoid this situation it is required the fact that X is a Banach space. Unfourtunately, the space $X = \{v \in \mathcal{C}^2[0,\pi] : v(0) = v(\pi) = 0\}$ equipped with the norm (2.3) is not a Banach space, namely there exists a Cauchy sequence of elements of X that does not converge to an element of X.

Exercise 18. *Prove that $X = \{v \in \mathcal{C}^2[0,\pi] : v(0) = v(\pi) = 0\}$ equipped with the norm (2.3) is not a Banach space.*

We will come back to these kind of results in the last chapter, but firstly here are some necessary mathematical background and preliminary theory.

2.2 Brief reminder of Lebesgue spaces

Given a natural number $N \in \mathbb{N}$, $p \in [1, \infty]$ and a non-empty and open subset Ω of $\mathbb{R}^N$, the Lebesgue space $L^p(\Omega)$ is the following vector space

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable } / \int_\Omega |f(x)|^p\,dx < \infty\} \quad (1 \leq p < \infty)$$

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable } / \ \exists C \geq 0 : |f(x)| \leq C \text{ a.e. on } \Omega\}$$

Lebesgue spaces can be equipped with the norms

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} \quad \forall f \in L^p(\Omega) \quad (1 \le p < \infty)$$
$$\|f\|_{L^\infty(\Omega)} = \inf\left\{C \in \mathbb{R}_0^+ : |f(x)| \le C \text{ a.e. on } \Omega\right\} \quad \forall f \in L^\infty(\Omega)$$

The definitions from above are norms on the corresponding Lebesgue spaces and this follows from the Hölder's inequality. Several important properties of Sobolev spaces are stated now and their proof can be read in [8]. Firstly, let us recall some definitions concerning functional spaces.

Definition 2 (Reflexive space). *A Banach space E is **reflexive** when the canonical injection $J : E \to E^{**}$ from the space E into its bidual space E^{**} (see [8]) is surjective, i.e. $J(E) = E^{**}$ (E is identified with E^{**}).*

Definition 3 (Separable space). *A metric space E is **separable** when there exists a countable and dense subset of E.*

Proposition 11 (Basic properties of Lebesgue spaces). *Let $m \in \mathbb{N}$ be a natural number and $\Omega \subset \mathbb{R}$ a non-empty and open subset of $\mathbb{R}$, then*

(a) $L^p(\Omega)$ is a Banach space for $1 \le p \le \infty$.

(b) $L^p(\Omega)$ is a reflexive space for $1 < p < \infty$.

(c) $L^p(\Omega)$ is a separable space for $1 \le p < \infty$.

(d) $L^2(\Omega)$ is a Hilbert space with the following scalar product

$$\langle f, g\rangle_{L^2} = \langle f, g\rangle_{L^2} = \int_\Omega f(x)g(x) \, dx \quad \forall f, g \in L^2(\Omega)$$

Given a non-empty and open subset Ω of $\mathbb{R}^N$, we wonder if there is a relationship between $L^p(\Omega)$ and $L^q(\Omega)$. Yes, there is but we have to distinguish two cases.

Case 1. The measure $\mu(\Omega)$ of Ω is finite.
If $1 \le p < q \le \infty$, then $L^q(\Omega) \subset L^p(\Omega)$ and the inclusion is continuous. For any $f \in L^q(\Omega)$, applying Hölder's inequality with exponents $q/(q-p)$ and q/p,

$$\begin{aligned}
\|f\|_{L^p(\Omega)} &= \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} = \left(\int_\Omega 1 \cdot |f(x)|^p \, dx\right)^{1/p} \le \\
&\le \left[\left(\int_\Omega 1 \, dx\right)^{\frac{q-p}{q}} \left(\int_\Omega (|f(x)|^p)^{q/p} \, dx\right)^{\frac{p}{q}}\right]^{1/p} = \\
&= \left[\left(\int_\Omega 1 \, dx\right)^{\frac{q-p}{pq}}\right]\left[\left(\int_\Omega |f(x)|^q \, dx\right)^{\frac{1}{q}}\right] = \mu(\Omega)^{\frac{q-p}{pq}} \|f\|_{L^q(\Omega)} < \infty
\end{aligned}$$

Case 2. The measure $\mu(\Omega)$ of Ω is infinite.
In this case there is no relationship by far and it is not hard to think in a counterxample, at least, in one dimension. For any values p and q, there exist functions u and v such that $u \in L^p(\Omega)$ but $u \notin L^q(\Omega)$ and $v \in L^q(\Omega)$ but $v \notin L^p(\Omega)$.

So, in general, for an arbitrary subset $\Omega \subset \mathbb{R}^N$ (of finite or infinite measure), one cannot conclude anything about the relationship among the Lebesgue spaces. However, if a function belongs to two Lebesegue spaces of exponents p and q with $p < q$, then it must belong to all the Lebesgue spaces with exponents in $[p, q]$. This is the same as saying that, for a given function $u : \Omega \to \mathbb{R}$, the set $\{p \in [1, \infty] : u \in L^p(\Omega)\}$ is an interval. Of course, it could be the hole $[1, \infty]$ or the empty set or just one point. This is called the *theorem of interpolation of Lebesgue spaces* and is proved next.

Theorem 10 (Interpolation of Lebesgue spaces)**.** *Let $\Omega \subset \mathbb{R}^N$ a non-empty and open subset of $\mathbb{R}^N$ and $1 \leq p \leq l \leq q < \infty$, then $L^p(\Omega) \cap L^q(\Omega) \subset L^l(\Omega)$. In addition, if $f \in L^p(\Omega) \cap L^q(\Omega)$, then*

$$\|f\|_{L^l(\Omega)} \leq \|f\|_{L^p(\Omega)}^{\frac{p(q-l)}{l(q-p)}} \cdot \|f\|_{L^q(\Omega)}^{\frac{q(l-p)}{l(q-p)}}$$

Proof. We are going to apply Hölder's inequality with the exponents p/α and q/β where

$$\alpha = \frac{p(q-l)}{q-p} \quad \text{and} \quad \beta = \frac{q(l-p)}{q-p}$$

Note that

$$\alpha + \beta = l \text{ and } \frac{\alpha}{p} + \frac{\beta}{q} = 1$$

We compute the following integral for any $f \in L^p(\Omega) \cap L^q(\Omega)$ and any $l \in [p, q]$. Clearly, $f^\alpha \in L^{\frac{p}{\alpha}}(\Omega)$ and $f^\beta \in L^{\frac{q}{\beta}}(\Omega)$,

$$\begin{aligned}
\|f\|_{L^l(\Omega)}^l &= \int_\Omega |f(x)|^l \, dx = \int_\Omega |f(x)|^\alpha |f(x)|^\beta \, dx \leq \\
&\leq \left(\int_\Omega (|f(x)|^\alpha)^{p/\alpha} \, dx\right)^{\alpha/p} \left(\int_\Omega (|f(x)|^\beta)^{q/\beta} \, dx\right)^{\beta/q} = \\
&= \left(\int_\Omega |f(x)|^p \, dx\right)^{\alpha/p} \left(\int_\Omega |f(x)|^q \, dx\right)^{\beta/q} = \\
&= \|f\|_{L^p(\Omega)}^\alpha \|f\|_{L^q(\Omega)}^\beta < \infty
\end{aligned}$$

this computation shows that $f \in L^l(\Omega)$ and also is the required inequality. □

Exercise 19. *Take some arbitrary interval I and find a function $f : \Omega = (0, \infty) \to \mathbb{R}$ such that $f \in L^p(\Omega)$ if, and only if, $p \in I$.*

2.3 Introduction to Sobolev spaces

2.3.1 Sobolev spaces in one dimension

Proposition 12. *Let $I \subset \mathbb{R}$ be a non-empty and open interval of $\mathbb{R}$ and two real-valued functions $u \in \mathcal{C}^1(I)$ and $f \in \mathcal{C}^0(I)$, then the following two statements are equivalent:*

(i) $u' = f$

(ii) $\displaystyle\int_I u(x)v'(x)\ dx = -\int_I f(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(I)$

Proof.

$i) \Rightarrow ii)$ Given an arbirary $v \in \mathcal{C}_0^\infty(I)$, since v has compact support on I, there exists a compact interval $]a, b[\subset I$ such that $v(x) = 0$ for all $x \in I \backslash]a, b[$ and so, by integration by parts formula,

$$\begin{aligned}
\int_I u(x)v'(x)\ dx &= \int_a^b u(x)v'(x)\ dx = u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\ dx = \\
&= -\int_a^b u'(x)v(x)\ dx = -\int_a^b f(x)v(x)\ dx = -\int_I f(x)v(x)\ dx
\end{aligned}$$

$ii) \Rightarrow i)$ Given an arbirary $v \in \mathcal{C}_0^\infty(I)$, again by integration by parts formula

$$\begin{aligned}
\int_I (u' - f)(x)v(x)\ dx &= \int_I u'(x)v(x)\ dx - \int_I f(x)v(x)\ dx = \\
&= \int_a^b u'(x)v(x)\ dx - \int_a^b f(x)v(x)\ dx = \\
&= -\int_a^b u(x)v'(x)\ dx - \int_a^b f(x)v(x)\ dx = 0
\end{aligned}$$

Due to $u' - f$ is a continuous function, the fact that $\int_I (u' - f)(x)v(x)\ dx = 0$ for each $v \in \mathcal{C}_0^\infty(I)$ implies that $u' - f = 0$ in I.

□

This proposition leads to introduce the *weak derivation.* For that aim, it is useful to recall the following property:

Exercise 20. *If $I \subset \mathbb{R}$ is a non-empty and open interval, prove that*

$$\begin{aligned}
L^1_{loc}(I) &\overset{def}{=} \{u : I \to \mathbb{R} \mid \forall x_* \in I,\ \exists \rho > 0 : u \in L^1(I \cap [x_* - \delta, x_* + \delta])\} = \\
&= \{u : I \to \mathbb{R} \mid u \in L^1(K),\ \forall K \subset I \ \mathit{compact}\}
\end{aligned}$$

Definition 4 (Weak derivation)**.** *Given $I \subset \mathbb{R}$ a non-empty and open interval and $u, f \in L^1_{loc}(I)$, f is the **weak derivative** of u when*

$$\int_I u(x)v'(x)\ dx = -\int_I f(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(I)$$

Remark 9. *If a function is classically derivable, then it is also weakly derivable and both derivatives coincide. This is nothing but the integration by parts theorem.*

Remark 10. *Given a non-empty and open interval $I \subset \mathbb{R}$, if a function $u \in L^1_{loc}(I)$ is weakly derivable, then its weak derivative is unique. This follows easily from the fundamental lemma of Calculus of variations.*

Example 5. *In this example, we calculate the weak derivative of the function $u :]-1,1[\to \mathbb{R}$ given by $u(x) = |x|$ for all $x \in (-1,1)$ which clearly is not classically derivable at the origin, but yes it is weakly derivable as it is shown next. Let us define the function $f :]-1,1[\to \mathbb{R}$ by*

$$f(x) = sgn(x) = \begin{cases} -1 & si\ -1 < x < 0 \\ 0 & si\ x = 0 \\ 1 & si\ 0 < x < 1 \end{cases}$$

We claim that $u' = f$ (in the weak sense, obviously). In order to prove this, we need to show that

$$\int_{-1}^1 |x|v'(x)\ dx = -\int_{-1}^1 sgn(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(]-1,1[)$$

Let v be an arbitrary function in $\mathcal{C}_0^\infty(]-1,1[)$, then we write the following

$$\begin{aligned}
\int_{-1}^1 |x|v'(x)\ dx &= \int_{-1}^0 -xv'(x)\ dx + \int_0^1 xv'(x)\ dx = \\
&= -xv(x)]_{-1}^0 + \int_{-1}^0 v(x)\ dx + xv(x)]_0^1 - \int_{-1}^0 v(x)\ dx = \\
&= \int_{-1}^0 v(x)\ dx - \int_0^1 v(x)\ dx = -\int_{-1}^1 sgn(x)v(x)\ dx
\end{aligned}$$

Example 6. *It is reasonable to think that the possible weak derivative of the sign function is the zero-constant function. In this example, we show that the weak derivative of the sign function is not the zero function. Let v be an arbitrary function in $\mathcal{C}_0^\infty(]-1,1[)$, then by Barrow's rule*

$$\int_{-1}^1 sgn(x)v'(x)\ dx = \int_{-1}^0 -v'(x)\ dx + \int_0^1 v'(x)\ dx = -2v(0)$$

and $-2v(0)$ needs not to be zero for all $v \in \mathcal{C}_0^\infty(]-1,1[)$.

The previous example only shows that zero function is not the weak derivative of the sign function on $]-1,1[$. the question is: is really the sign function weakly derivable? The answer is no and the next proposition will help us to understand why. Its proof follows from the definition of weak derivative.

Proposition 13 (Local property of weak derivative)**.** *Let be $I, J \subset \mathbb{R}$ two non-empty and open subsets of $\mathbb{R}$ with $J \subset I$ and $u : I \to \mathbb{R}$ a weakly derivable function on I, then the restriction $u|_J : J \to \mathbb{R}$ is also weakly derivable with $(u|_J)' = u'|_J$.*

How can be this proposition used in order to see that the sign function has no weak derivative? Suppose that $u = sgn$ is weakly derivable on $]-1,1[$, then

$$(u|_{]-1,0[})' = f|_{]-1,0[} \Rightarrow (-1)' = f|_{]-1,0[} \Rightarrow f|_{]-1,0[} \equiv 0 \text{ a.e. on }]-1,0[$$

$$(u|_{]0,1[})' = f|_{]0,1[} \Rightarrow (+1)' = f|_{]0,1[} \Rightarrow f|_{]0,1[} \equiv 0 \text{ a.e. on }]0,1[$$

So, if u was weakly derivable, its weak derivative $f = u'$ would be zero but this contradicts the example 6.

Notation. From now on, $W^1(I)$ will denote the set of functions $u : I \to \mathbb{R}$ that are weakly derivable on I. By induction, one can define $W^m(I) = \left\{u \in W^1(I) : u' \in W^{k-1}(I)\right\}$ for each natural number $m \geq 2$, in other words $W^m(I)$ is the set of m-times weakly derivable functions. Note that $W^m(I)$ is a vector space, that is, the addition of two functions u and v of $W^m(I)$ belongs to $W^m(I)$ with $(u+v)' = u' + v'$ on I and the multiplication of a function u of $W^m(I)$ by a scalar $k \in \mathbb{R}$ belongs to $W^m(I)$ with $(ku)' = ku'$ on I.

It is known that if a function u is derivable on a non-empty and open interval I (in the classical sense) and its derivative u' is constantly zero, then u is constant on I. An analogous result is fulfilled for weak derivative.

Proposition 14. *Let $I \subset \mathbb{R}$ be a non-empty and open interval and $u \in W^1(I)$ such that $u' = 0$ a.e. on I, then there exists $K \in \mathbb{R}$ with $u(x) = K$ at almost every point $x \in I$.*

Proof. By definition, $u \in L^1_{loc}(I)$ and

$$\int_I u(x)v'(x)\ dx = -\int_I 0 \cdot v(x)\ dx = 0 \quad \forall v \in \mathcal{C}_0^\infty(I)$$

Take a function $\psi \in \mathcal{C}_0^\infty(I)$ with $\int_I \psi(x)\ dx = 1$ and for every $\phi \in \mathcal{C}_0^\infty(I)$ we write,

$$\phi(x) = \underbrace{\phi(x) - \int_I \phi(x)\ dx \cdot \psi(x)}_{\alpha(x)} + \underbrace{\int_I \phi(x)\ dx \cdot \psi(x)}_{\beta(x)} \quad \forall x \in I$$

Clearly, $\alpha, \beta \in \mathcal{C}_0^\infty(I)$, $\int_I \alpha(x)\ dx = 0$ and there exists an interval $]a,b[\subset I$ such that $\alpha(x) = 0$ for all $x \in I \backslash]a,b[$. Consider the function $v : I \to \mathbb{R}$ defined as

$$v(x) = \int_a^x \alpha(t)\ dt \quad \forall x \in I$$

Since $v' = \alpha$, one has that $v \in \mathcal{C}^\infty(I)$ and also v has compact support on I:

$$x \leq a \Rightarrow \alpha = 0 \Rightarrow v(x) = 0$$

$$x \geq b \Rightarrow v(x) = \underbrace{\int_a^b \alpha(t)\ dt}_{=\int_I \alpha(t)\ dt = 0} + \int_b^x \underbrace{\alpha(t)}_{=0}\ dt = 0$$

Overall,

$$\int_I u(x)v'(x)\ dx = \int_I u(x)\alpha(x)\ dx = 0$$

and let us calculate the quantity

$$\gamma = \int_I \left(u(x) - \int_I u(t)\psi(t)\ dt \right) \phi(x)\ dx$$

Since $\alpha = \phi - (\int_I \phi)\psi$, one may write for every $\phi \in \mathcal{C}_0^\infty(I)$

$$\begin{aligned}
\gamma &= \int_I \left(u(x) - \int_I u(t)\psi(t)\ dt \right) \left(\alpha(x) + \int_I \phi(t)\ dt \cdot \psi(x) \right)\ dx = \\
&= \underbrace{\int_I u(x)\alpha(x)\ dx}_{0} + \int_I \phi(t)\ dt \cdot \int_I u(x)\psi(x)\ dx - \\
&- \int_I u(t)\psi(t)\ dt \cdot \underbrace{\int_I \alpha(x)\ dx}_{0} - \int_I u(t)\psi(t)\ dt \cdot \int_I \phi(t)\ dt \cdot \underbrace{\int_I \psi(x)\ dx}_{1} = 0
\end{aligned}$$

So, we get that

$$\int_I \left(u(x) - \int_I u(t)\psi(t)\ dt \right) \phi(x)\ dx = 0 \quad \forall \phi \in \mathcal{C}_0^\infty(I)$$

which implies that

$$u(x) = \int_I u(t)\psi(t)\ dt \ \text{ a.e. on } I$$

□

The next result is interesting itself, it asserts that every weakly derivable function u on a non-empty and open interval I is continuous on it, actually. This must be understood as there exists a continuous function $\overline{u}$ on I such that $u = \overline{u}$ a.e. on I, namely there is one and only one continuous representative on I of u. Moreover, the Barrow's rule holds true as it is shown now.

Proposition 15. *Let $I \subset \mathbb{R}$ be a non-empty and open interval and $u \in W^1(I)$, then there exists a unique $\overline{u} \in \mathcal{C}^0(I)$ such that $u(x) = \overline{u}(x)$ for almost all point of I. Futhermore, the next equality holds for every pair of points $\alpha, \beta \in I$*

$$\int_\alpha^\beta u'(t)\ dt = \overline{u}(\beta) - \overline{u}(\alpha)$$

Remark 11. *Note that in the previous proposition it does not make sense to write*

$$\int_\alpha^\beta u'(t)\ dt = u(\beta) - u(\alpha)$$

because u is a locally integrable function on I and it is not defined pointwise on I. One can change the value of u at the points α and β and still have the same function!

Proof. Take a fixed point x_0 of I and define $\phi = \int_{x_0}^x u'$. By the fundamental theorem of calculus, ϕ is a continuous function of I and $\phi' = u' \in L^1_{loc}(I)$ by the definition of weak derivative. Next, it is proved that

$$\int_I \phi(x)v'(x)\ dx = -\int_I u'(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(I)$$

Assume that $v = 0$ on $I\backslash]a, b[$, so

$$\begin{aligned}
\int_I \phi(x)v'(x)\ dx &= \int_a^b \phi(x)v'(x)\ dx = \int_a^{x_0} \phi(x)v'(x)\ dx + \int_{x_0}^b \phi(x)v'(x)\ dx = \\
&= \int_a^{x_0}\left(\int_{x_0}^x u'(t)\ dt\right)v'(x)\ dx + \int_{x_0}^b\left(\int_{x_0}^x u'(t)\ dt\right)v'(x)\ dx = \\
&= \underbrace{-\int_a^{x_0}\left(\int_x^{x_0} u'(t)\ dt\right)v'(x)\ dx}_{J_1} + \underbrace{\int_{x_0}^b\left(\int_{x_0}^x u'(t)\ dt\right)v'(x)\ dx}_{J_2}
\end{aligned}$$

Applying Fubini's theorem,

$$\begin{aligned}
J_1 &= \int_a^{x_0}\left(\int_x^{x_0} u'(t)v'(x)\ dt\right)\ dx = \int_a^{x_0}\left(\int_a^t u'(t)v'(x)\ dx\right)\ dt = \\
&= \int_a^{x_0} u'(t)[v(t) - \underbrace{v(a)}_{0}]\ dt = \int_a^{x_0} u'(t)v(t)\ dt \\
J_2 &= \int_{x_0}^b\left(\int_{x_0}^x u'(t)v'(x)\ dt\right)\ dx = \int_{x_0}^b\left(\int_t^b u'(t)v'(x)\ dx\right)\ dt = \\
&= \int_{x_0}^b u'(t)[\underbrace{v(b)}_{0} - v(t)]\ dt = -\int_{x_0}^b u'(t)v(t)\ dt
\end{aligned}$$

Consequently,

$$\int_I \phi(x)v'(x)\,dx = -\int_a^{x_0} u'(t)v(t)\,dt - \int_{x_0}^b u'(t)v(t)\,dt = -\int_a^b u'(t)v(t)\,dt = -\int_I u'(x)v(x)\,dx$$

So, $\phi' = u'$, therefore $(\phi - u)' = 0$ and thus, by the proposition 14, there exists $K \in \mathbb{R}$ such that $u = \phi + K$ a.e. on I. Set $\overline{u}(x) = \phi(x) + K$ for all $x \in I$, then $\overline{u}$ is continuous on I and of course $\overline{u} = u$ a.e. on I. Finally,

$$\overline{u}(\beta) - \overline{u}(\alpha) = \phi(\beta) - \phi(\alpha) = \int_{x_0}^{\beta} u'(t)\,dt - \int_{x_0}^{\alpha} u'(t)\,dt = \int_{\alpha}^{\beta} u'(t)\,dt$$

As for the uniqueness of $\overline{u}$, assume that there are two continuous functions $\overline{u}_1$ and $\overline{u}_2$ on I such that $u = \overline{u}_1$ and $u = \overline{u}_2$ a.e. on I, then $\overline{u}_1 = \overline{u}_2$ a.e. on I, but both functions are continuous so the only possibility left is that $\overline{u}_1(x) = \overline{u}_2(x)$ for all $x \in I$. □

Remark 12. *The previous definitions and results still hold when the open interval I is replaced by an arbitrary open subset Ω of $\mathbb{R}$ which is nothing but a countable union of open and disjoint intervals of $\mathbb{R}$.*

Definition 5 (Sobolev spaces)**.** *Given a non-empty and open subset Ω of $\mathbb{R}$, $m \in \mathbb{N}$ and $1 \le p \le \infty$, the* ***Sobolev space*** *$W^{m,p}(\Omega)$ is the following functional space*

$$W^{m,p}(\Omega) = \left\{u \in W^m(\Omega) : u^{(k)} \in L^p(\Omega), k = 1, ..., m\right\}$$

where $u^{(k)}$ denotes the k-th weak derivative of u from $k = 1$ to $k = m$.

Sobolev spaces newly introduced are richer spaces than $W^m(\Omega)$, meaning that Sobolev spaces are more than vector spaces. For example, they are normed spaces with these norms below for each $u \in W^{m,p}(\Omega)$

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{k=1}^{m} \|u^{(k)}\|_{L^p}^p\right)^{1/p} = \left(\int_\Omega |u(x)|^p + \cdots + |u^{(m)}(x)|^p\,dx\right)^{1/p} \quad (1 \le p < \infty)$$

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{k=1,\ldots,m} \left\{\|u^{(k)}\|_{L^\infty}\right\}$$

Several important properties of Sobolev spaces are stated now and their proof can be read in [8]. Soon it will become evident the significance of these properties that Sobolev spaces enjoy.

Proposition 16 (Basic properties of Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}$ be a non-empty and open subset of $\mathbb{R}$ and $m \in \mathbb{N}$, then*

(a) $W^{m,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$.

(b) $W^{m,p}(\Omega)$ is a reflexive space for $1 < p < \infty$.

(c) $W^{m,p}(\Omega)$ is a separable space for $1 \le p < \infty$.

(d) $W^{m,2}(\Omega) := H^m(\Omega)$ is a Hilbert space with the following scalar product

$$\langle u, v\rangle_{H^m} = \sum_{k=1}^{m} \langle u^{(k)}, v^{(k)}\rangle_{L^2} = \int_\Omega u(x)v(x)\,dx + \cdots + \int_\Omega u^{(m)}(x)v^{(m)}(x)\,dx \quad \forall u, v \in H^m(\Omega)$$

2.3.2 Sobolev spaces in several dimensions

Henceforth, $N \in \mathbb{N}$ will be an arbitrary natural number. Let us start considering functions of several variables. The following proposition is the N-dimensional version of the proposition 12 but before proving it, it will help to remember the *divergence theorem* on which the proof is based.

Divergence theorem. If $\Omega \subset \mathbb{R}^N$ is a non-empty and bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $F = (F_1, ..., F_N) : \Omega \to \mathbb{R}^N$ a $\mathcal{C}^1$-vector field on Ω, then

$$\int_\Omega \operatorname{div} F(x)\ dx = \int_{\partial\Omega} \langle F, \eta\rangle(x)\ d\sigma(x)$$

where η is the outward pointing unit normal field of the boundary. Recall that the divergence of F is defined by

$$\operatorname{div} F(x) = \sum_{k=1}^{N} \frac{\partial F_k}{\partial x_k}(x) \quad \forall x \in \Omega$$

Proposition 17. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty and bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and two real-valued functions $u \in \mathcal{C}^1(\Omega)$ and $f \in \mathcal{C}^0(\Omega)$ and an index $k \in \{1, ..., N\}$, then the following two statements are equivalent:*

(i) $\dfrac{\partial u}{\partial x_k} = f$

(ii) $\displaystyle\int_I u(x) \frac{\partial v}{\partial x_k}(x)\ dx = -\int_I f(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$

Proof.

$i) \Rightarrow ii)$ It is enough to apply the divergence theorem with $F : \Omega \to \mathbb{R}$ given by

$$F(x) = (0, ..., 0, \underbrace{u(x)v(x)}_{k)-\text{position}}, 0, ..., 0) \quad \forall x \in \Omega$$

Since $v = 0$ on $\partial\Omega$, it follows $F = (0, ..., 0)$ on $\partial\Omega$, so the divergence theorem gives the following equality

$$0 = \int_\Omega \operatorname{div} F(x)\ dx = \int_\Omega \frac{\partial uv}{\partial x_k}(x)\ dx = \int_\Omega \left(\frac{\partial u(x)}{\partial x_k} v(x) + u(x) \frac{\partial v(x)}{\partial x_k} \right)\ dx$$

and hence,

$$\int_\Omega u(x) \frac{\partial v(x)}{\partial x_k}\ dx = -\int_\Omega \frac{\partial u(x)}{\partial x_k} v(x)\ dx = -\int_\Omega f(x)v(x)\ dx$$

ii) $\Rightarrow$ *i)* Again, for every $v \in \mathcal{C}_0^\infty(\Omega)$, by the divergence theorem

$$\int_\Omega u(x)\frac{\partial v(x)}{\partial x_k}\ dx = -\int_\Omega \frac{\partial u(x)}{\partial x_k} v(x)\ dx$$

and, by hypothesis

$$\int_\Omega u(x)\frac{\partial v(x)}{\partial x_k}\ dx = -\int_\Omega f(x)v(x)\ dx$$

from these two equalities, one gets the following

$$\int_\Omega \left(\frac{\partial u(x)}{\partial x_k} - f(x)\right) v(x)\ dx = 0 \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$$

and, by the continuity of $\dfrac{\partial u}{\partial x_k} - f$, this leads to $f(x) = \dfrac{\partial u(x)}{\partial x_k} \quad \forall x \in \Omega.$

$\square$

Once we have introduced the weak derivative in several variables it is natural to wonder if every weak derivable function is also continuous, as happens with the classical derivability. The answer is no, except in one dimension (take a look at proposition 15). To see this, set $B = B(0,1)$ the unit open ball in $\mathbb{R}^N$, $\alpha \in \mathbb{R}$ and the function $u_\alpha(x) = \|x\|^{-\alpha}$ on B (note that u_α might have problems when it is evaluated at zero, but it does not matter because it is just one point). We want to answer the question: for which values of α, the function u_α belongs to $W^1(B)$? Since $u_\alpha(x) = (x_1^2 + \cdots + x_N^2)^{-\alpha/2}$, the natural candidate to the weak derivative of u_α would be

$$\frac{\partial u_\alpha}{\partial x_k}(x) = -\frac{\alpha}{2}(x_1^2 + \cdots + x_N^2)^{-\alpha/2-1}2x_k = -\alpha x_k\|x\|^{-\alpha-2} \quad \text{a.e. on } B \qquad (k = 1, ..., N) \tag{2.4}$$

In view of the proposition 13 and the fact that $(u_\alpha)|_{B\setminus\overline{B}(0,\varepsilon)}$ is classically derivable, then its weak derivative must coincide with its classical one which is nothing but the above computation; therefore either the weak derivative of u_α is (2.4) for $k = 1, ..., N$ or does not exist. The first thing to check is u_α and the possible candidate $\partial u_\alpha/\partial x_k$ are $L^1_{loc}(B)$-functions. The only problematic point here is the origin and so, this is equivalent to check that

$$I = \int_B \frac{1}{\|x\|^\alpha}\ dx < \infty \quad \text{and} \quad J = \int_B \frac{\alpha|x_k|}{\|x\|^{\alpha+2}} < \infty$$

Note that

$$\begin{aligned} I &= \int_B \frac{1}{\|x\|^\alpha}\ dx = \int_0^1 \left(\int_{|x|=r} \frac{1}{\|x\|^\alpha}\ d\sigma\right) dr = \\ &= \omega_N \textstyle\int_0^1 r^{N-1-\alpha}\ dr = \begin{cases} \omega_N \left.\frac{r^{N-\alpha}}{N-\alpha}\right]_0^1 \\ \omega_N \log(r)]_0^1 \end{cases} \end{aligned}$$

where ω_N is the measure of the $(N-1)$-dimensional sphere, and so $I < \infty$ if, and only if, $\alpha < N$. In respect for J, obviously

$$J < \infty \Leftrightarrow \int_B \frac{|x_k|}{\|x\|^{\alpha+2}}\ dx < \infty$$

and

$$\int_B \frac{|x_k|}{\|x\|^{\alpha+2}}\ dx \leq \int_B \frac{\|x\|}{\|x\|^{\alpha+2}}\ dx = \int_B \frac{1}{\|x\|^{\alpha+1}}\ dx$$

and again if $\alpha < N-1$ then J is finite, thus taking $\alpha < N-1$ both I and J are finite. At this point, there is just one thing left to check to prove that 2.4 is the weak derivative of u_α, actually. We need to check that the following formula holds for all $v \in \mathcal{C}_0^\infty(B)$

$$\int_B \frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k}\ dx = -\int_B -\frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x)\ dx$$

Equivalently,

$$\int_B \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x)\right)\ dx = 0 \quad \forall v \in \mathcal{C}_0^\infty(B)$$

In order to check that, we fix $0 < \varepsilon < 1$ and write the previous integral as a sum of these two

$$\underbrace{\int_{B\setminus\overline{B}(0,\varepsilon)} \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x)\right)\ dx}_{A} \quad \text{and} \quad \underbrace{\int_{B(0,\varepsilon)} \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x)\right)\ dx}_{B}$$

Let us discuss A firstly. By the divergence theorem with $F(x) = (0, ..., 0, \frac{1}{\|x\|^\alpha} v(x), 0, ..., 0)$, which can be applied since 0 is not in $B \setminus \overline{B}(0,\varepsilon)$, we have

$$\begin{aligned} A &= \int_{B\setminus\overline{B}(0,\varepsilon)} \frac{\partial}{\partial x_k}\left(\frac{1}{\|x\|^\alpha} v(x)\right)\ dx = \int_{B\setminus\overline{B}(0,\varepsilon)} \operatorname{div} F(x)\ dx = \int_{\partial B\setminus\overline{B}(0,\varepsilon)} \langle F, \eta_e\rangle(x)\ dx = \\ &= \int_{\|x\|=1} \langle F, \eta_e\rangle(x)\ dx + \int_{\|x\|=\varepsilon} \langle F, \eta_e\rangle(x)\ dx \end{aligned}$$

where η_e is the outward pointing unit normal field to the corresponding sphere, but v has compact support on B, so $v(x) = 0$ when $\|x\| = 1$ and $F(x) = 0$ when $\|x\| = 1$, hence the integral over the sphere of radius 1 turns out to be zero. What about the other one? Well, using the definition of F and the fact that

$$\frac{|x_k|}{\varepsilon} \leq \frac{\|x\|}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = 1 \quad \forall x$$

on the sphere of radius ε and the fact that there exists $M > 0$ such that

$$|v(x)| \leq M$$

on the sphere of radius ε by the compactness of the sphere, it is clear that

$$\begin{aligned}
\int_{\|x\|=\varepsilon} \langle F, \eta_e \rangle(x)\, d\sigma &= \int_{\|x\|=\varepsilon} -\frac{x_k}{\varepsilon}\frac{v(x)}{\varepsilon^\alpha}\, d\sigma \leq \\
&\leq \int_{\|x\|=\varepsilon} \frac{|x_k|}{\varepsilon}\frac{|v(x)|}{\varepsilon^\alpha}\, d\sigma \leq \\
&\leq \int_{\|x\|=\varepsilon} \frac{|v(x)|}{\varepsilon^\alpha}\, d\sigma \leq \\
&\leq M \int_{\|x\|=\varepsilon} \frac{1}{\varepsilon^\alpha}\, dx = \\
&= \frac{M}{\varepsilon^\alpha}\mu(\|x\| = \varepsilon) = \frac{M}{\varepsilon^\alpha}\omega_N \varepsilon^{N-1} = M\omega_N\varepsilon^{N-1-\alpha} \to 0 \ (\varepsilon \to 0)
\end{aligned}$$

due to $\alpha < N-1$. Now, it is turn to discuss B: by the dominated convergence theorem one may write

$$\int_{B(0,\varepsilon)} \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x) \right) dx = \int_B \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x) \right) \chi_{B(0,\varepsilon)}(x)\, dx$$

where $\chi_{B(0,\varepsilon)}$ is such that $\chi_{B(0,\varepsilon)} = 1$ on $B(0,\varepsilon)$ and $\chi_{B(0,\varepsilon)} = 0$ on $\mathbb{R}^N \setminus B(0,\varepsilon)$, and then use that $\chi_{B(0,\varepsilon)} \to 0$ as $\varepsilon \to 0$.

Overall, we have just proved that

$$\int_{B(0,1)} \left(\frac{1}{\|x\|^\alpha}\frac{\partial v(x)}{\partial x_k} - \frac{\alpha x_k}{\|x\|^{\alpha+2}} v(x) \right) dx = \quad \forall v \in \mathcal{C}_0^\infty(B)$$

which means that $-\alpha x_k \|x\|^{-\alpha-2}$ is de weak derivative with respect to x_k of u provided that $\alpha < N-1$. Finally, we are able to keep the promise of solving the question *weak derivability* $\Rightarrow$ *continuity*? No, just take $N \geq 2$ and $0 < \alpha < N-1$, then the function u_α is weakly derivable but it is not continuous at zero.

High order weak partial derivatives. A *multi-index* $\alpha = (\alpha_1, ..., \alpha_N)$ is a N-tuple with $\alpha_k \in \mathbb{N} \cup \{0\}$ for all $k = 1, ..., N$. The *order* of a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ is often denoted by $|\alpha|$ and defined as $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Given a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$, $\Omega \subseteq \mathbb{R}^N$ a non-empty and open subset of $\mathbb{R}^N$ and $u \in \mathcal{C}^\infty(\Omega)$ is a smooth function on Ω, the α-**classical derivative** of u is denoted by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

that is, u is derivated (classically) α_k times with respect to the variable x_k for each $k = 1, ..., N$. Similarly, the α-**weak derivative** of u is denoted by

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

that is, u is derivated (weakly) α_k times with respect to the variable x_k for each $k = 1, ..., N$. Using the same notation as in one dimension, for every $m \in \mathbb{N}$, $W^m(\Omega)$ will denote the set of functions $u : \Omega \to \mathbb{R}$ that are weakly derivable up to order m on Ω, meaning that the α-weak derivative $\partial^\alpha u$ of u exists for all multi-index α with $|\alpha| \leq m$. Basically, there are two ways to introduce the high order weak partial derivatives of a function. Note that $W^m(\Omega)$ is a vector space, that is, given a multi-idnex α of order lower or equal to m, the addition of two functions u and v of $W^m(\Omega)$ belongs to $W^m(\Omega)$ with $\partial^\alpha(u + v) = \partial^\alpha u + \partial^\alpha v$ on Ω and the multiplication of a function u of $W^m(\Omega)$ by a scalar $k \in \mathbb{R}$ belongs to $W^m(\Omega)$ with $\partial^\alpha(ku) = k\partial^\alpha u$ on Ω.

The α-weak derivative of u

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

is nothing but the α_N-times weak derivative with respect to x_N of the function

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_{N-1}^{\alpha_{N-1}}}$$

and this last derivative is nothing but the α_{N-1}-times weak derivative with respect to x_{N-1} of the function

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_{N-2}^{\alpha_{N-2}}}$$

and so on...

Another way to introduce the high order weak partial derivatives of a function is the following: given a multi-index α and a non-empty and open subset $\Omega \subseteq \mathbb{R}^N$, a function $f \in L^1_{loc}(\Omega)$ is the α-weak derivative of a function $u \in L^1_{loc}(\Omega)$ ($\partial^\alpha u = f$) when

$$\int_\Omega u(x)\partial^\alpha v(x)\ dx = (-1)^{|\alpha|} \int_\Omega f(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$$

Definition 6 (Sobolev spaces in N-dimensions)**.** *Given a non-empty and open subset Ω of $\mathbb{R}^N$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, the **Sobolev space** $W^{m,p}(\Omega)$ is the following functional space*

$$W^{m,p}(\Omega) = \{u \in W^m(\Omega) : \partial^\alpha u \in L^p(\Omega), |\alpha| \leq p\}$$

where $\partial^\alpha u$ denotes the α-weak derivative of u for a multi-index $\alpha = (\alpha_1, ..., \alpha_N)$ with $|\alpha| \leq p$.

Sobolev spaces newly introduced are richer spaces than $W^m(\Omega)$, meaning that Sobolev spaces are more than vector spaces. For example, they are normed spaces with these norms below for each $u \in W^{m,p}(\Omega)$

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p}^p \right)^{1/p} \qquad (1 \leq p < \infty)$$

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \leq m} \{\|\partial^\alpha u\|_{L^\infty}\}$$

All the results proved by the weak derivative in dimension one are still true translated into an arbitary finite dimension, except proposition 15. Here is a really proving-difficult theorem by G. Meyers and J. Serrin whose proof can be found in [9].

Theorem 11 (Meyers-Serrin). *Let $\Omega \subset \mathbb{R}^N$ be a non-empty and open subset of $\mathbb{R}^N$, then $\mathcal{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for all $m \in \mathbb{N}$ and $1 \leq p \leq \infty$.*

Another classical result concerning Sobolev spaces is the following.

Theorem 12 (Sobolev, Gagliardo, Nirenberg). *Let be $1 \leq p < N$ and $p_* = Np/(N-p)$, then $W^{1,p}(\mathbb{R}^N) \subset L^{p_*}(\mathbb{R}^N)$ and there exists a constant $C > 0$ (depending on p and N) such that*

$$\|u\|_{L^{p_*}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

The proof of this theorem is in [8] and it is helpful to prove the known *Sobolev embeddings.* Before stating this, let us suggest why the exponent $p_* = \frac{Np}{N-p}$ appears on the theorem... or even better, let us show that if there exists a constant $C > 0$ and $q \in [1, \infty]$ with

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \forall u \in C_c^\infty(\mathbb{R}^N)$$

then necessarily $q = p_*$. This is done by a simple scaling argument. Fix any funtion $u \in C_c^\infty(\mathbb{R}^N)$ which does not vanish identically and plug $u_\lambda(x) = u(\lambda x)$ defined on $\mathbb{R}^N$ $(\lambda > 0)$ into the inequality from above to get

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\lambda^{1+N/q-N/p}\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \forall \lambda > 0$$

In order to make the quantity $C\lambda^{1+N/q-N/p}$ lower than a constant for all $\lambda > 0$, we need to force $1 + N/q - N/p$ to be zero, and so $q = p_*$.

Theorem 13 (Sobolev embeddings on $\mathbb{R}^N$). *Set $p \in [1, \infty)$, then*

1. *if $1 \leq p < N$, $W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in [p, p_*]$, where $p_* = \frac{Np}{N-p}$,*

2. *if $p = N$, $W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in [N, \infty)$,*

3. *if $p > N$, $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$,*

and with continuous injection in the three cases, which basically means that there exists a constant $C = C(p, q, N) > 0$ such that

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)} \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

since the injection map is linear. Obviously, the constant C might be not the same in all the three cases.

Exercise 21. *Prove that Sobolev embeddings on $\mathbb{R}^N$ are not compact.*

The next theorem is the version of the previous one but now on bounded sets of $\mathbb{R}^N$ with smooth boundary.

Theorem 14 (Rellich-Kondrachov)**.** *Set $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^N$ a bounded set with smooth boundary, then*

1. *if $1 \leq p < N$, $W^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in [p, p_*]$, where $p_* = \frac{Np}{N-p}$,*
2. *if $p = N$, $W^{1,N}(\Omega) \subset L^q(\Omega)$ for all $q \in [N, \infty)$,*
3. *if $p > N$, $W^{1,p}(\Omega) \subset C(\overline{\Omega})$,*

and with compact injection in the three cases (except the injection $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$), which basically means that every bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$ (or in $C(\overline{\Omega})$ for the third case).

Exercise 22. *Prove that the exponents q in Rellich-Kondrachov theorem are optimal.*

It is easy now to extend the Sobolev embedding theorem to the case $W^{m,p}(\mathbb{R}^N)$ where $m \in \mathbb{N}$.

Corollary 4. *Set $p \in [1, \infty)$ and $m \in \mathbb{N}$, then*

1. *if $p < N/m$, then $W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in [p, p^*]$,*
2. *if $p = N/m$, then $W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in [p, \infty]$,*
3. *if $p > N/m$, then $W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$,*

and all these injections are continuous.

2.3.3 The space $W_0^{m,p}$

Definition 7 (Space $W_0^{m,p}$)**.** *Given a natural number $m \in \mathbb{N}$, a real number $p \in [1, \infty]$ and a non-empty and open subset Ω of $\mathbb{R}^N$, the Sobolev space $W_0^{m,p}(\Omega)$ is de closure in $W^{m,p}(\Omega)$ of the test-functions $\mathcal{C}_0^\infty(\Omega)$ over Ω, that is*

$$W_0^{m,p}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{W^{m,p}(\Omega)}$$

Equivalently,

$$u \in W_0^{m,p}(\Omega) \Longleftrightarrow u \in W^{m,p}(\Omega) \text{ and } \exists \{u_n\} \subset \mathcal{C}_0^\infty(\Omega) : u_n \overset{W^{m,p}(\Omega)}{\longrightarrow} u$$

It makes no sense to define the space $W_0^{m,p}(\Omega)$ as set of the $W^{m,p}(\Omega)$-functions that vanish on the boundary of Ω, since $\partial\Omega$ has zero measure. When more regularity is assumed, then this is true as shown the following proposition (see [8]).

Proposition 18. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty and bounded subset of $\mathbb{R}^N$ and $u \in \mathcal{C}^1(\overline{\Omega})$, then*

$$u \in W_0^{m,p}(\Omega) \Longleftrightarrow u(x) = 0 \quad \forall x \in \partial\Omega$$

The space $W_0^{1,p}$ enjoys a really useful inequality named *Poincaré inequality.*

Proposition 19 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^N$ be a non-empty and bounded subset of $\mathbb{R}^N$ with smooth boundary and $p \in [1, \infty)$, then there exists a constant $C > 0$ (depending only on p and Ω) such that*

$$\int_\Omega |\nabla u(x)|^p \, dx \geq C \int_\Omega |u(x)|^p \, dx \quad \forall u \in W_0^{1,p}(\Omega)$$

Futhermore, the optimal constant C is known to be the first eigenvalue of the $-\Delta$-operator on Ω.

The proof of this result is in [8] and it is essential the bounbdness assumption of Ω as it is shown in the next exercise.

Exercise 23. *Prove that if there is no Poincaré inqueality on $\mathbb{R}$.*

Example 7 (Which is the best constant for the Poincaré inquality when $p = 2$ and $\Omega = I = (0, \pi)$?)**.** *It is known that*

$$\int_0^\pi |u'(x)|^2 \, dx \geq C \int_0^\pi |u(x)|^2 \, dx \quad \forall u \in H_0^1(I)$$

for some positive constant C. Of course, the inequality holds for positive constant lower than C. The question is which is the biggest constant that we can fix in the inequality?
Take $u \in \mathcal{C}_0^\infty(I)$ and define

$$\varphi(x) = \frac{u(x)}{\sin(x)} \quad \forall x \in I$$

Note that φ is well-defined at $x = 0$ and $x = \pi$ because of the compact closure of u on I, so $\varphi(0) = \varphi(\pi) = 0$. Note that

$$\begin{aligned}
\int_0^\pi |u'(x)|^2 \, dx &= \int_0^\pi |(\varphi(x)\sin(x))'|^2 \, dx = \\
&= \int_0^\pi (\varphi'(x)^2 \sin(x)^2 + 2\varphi(x)\varphi'(x)\sin(x)\cos(x) + \varphi(x)^2\cos(x)^2) \, dx
\end{aligned}$$

and compute the next integral by the integration by parts formula,

$$\int_0^\pi 2\varphi(x)\varphi'(x)\sin(x)\cos(x)\ dx = \underbrace{\sin(x)\cos(x)\varphi(x)^2\big]_0^\pi}_{0} - \int_0^\pi \varphi(x)^2[\cos^2(x)-\sin^2(x)]\ dx$$

so, plugging this into the first computation arises the following

$$\begin{aligned}\int_0^\pi |u'(x)|^2\ dx &= \int_0^\pi (\varphi'(x)^2\sin(x)^2 + 2\varphi(x)\varphi'(x)\sin(x)\cos(x) + \varphi(x)^2\cos(x)^2)\ dx = \\ &= \int_0^\pi \varphi'(x)^2\sin(x)^2\ dx + \int_0^\pi \varphi(x)^2\sin(x)^2\ dx = \\ &= \int_0^\pi \varphi'(x)^2\sin(x)^2\ dx + \int_0^\pi |u(x)|^2\ dx \geq \int_0^\pi |u(x)|^2\ dx\end{aligned}$$

From this it follows that $C = 1$ is valid, but is 1 the best option?. Yes, it is because there is a function $u_* \in H_0^1(I)$ where

$$\int_0^\pi |u_*'(x)|^2\ dx = \int_0^\pi |u_*(x)|^2\ dx$$

and this would be true if

$$\int_0^\pi \varphi'(x)^2\sin(x)^2\ dx = 0$$

so one can take φ to be constant to achive it. Let say $\varphi \equiv 1$, so $u_* = \sin$.

In general, it is hard to find the optimal constant in Poincaré inequality and only in few cases it is known explicitly. One of this cases is for the N-dimensional cube, $\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N) \subset \mathbb{R}^N$ whose optimal constant turns out to be

$$C = \pi^2\left(\frac{1}{(b_1-a_1)^2} + \cdots + \frac{1}{(b_N-a_N)^2}\right)$$

and the equality in the Poincaré inequlity with this constant holds for the function

$$u(x) = \sin\left(\frac{\pi(x_1-a_1)}{b_1-a_1}\right)\cdots\sin\left(\frac{\pi(x_N-a_N)}{b_N-a_N}\right)$$

The best constant to fix in the Poincaré-inequality concerns the first eigenvalue of the linear Dirichlet problem (see next section).

2.4 Eigenvalues of linear Dirichlet boundary value problem

Let $\Omega \subset \mathbb{R}^N$ be a non-empty domain (open and connected subset) of $\mathbb{R}^N$ and consider the following problem related to $\lambda \in \mathbb{R}$ and Ω

$$(P_\lambda)\ \begin{cases} -\Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

This problem is called *linear Dirichlet boundary value problem* on Ω and it is clear that $u = 0$ is a trivial solution to (P_λ) for any $\lambda \in \mathbb{R}$. The interesting question to set out is if there exist values $\lambda \in \mathbb{R}$ such that (P_λ) admits non-trivial solutions.

The **eigenvalues** of (P_λ) on Ω are the values $\lambda \in \mathbb{R}$ (if exist) for which the problem (P_λ) admits non-trivial solutions and the **eigenfunctions** of (P_λ) on Ω are precisely the non-trivial solutions of (P_λ) associated to a eigenvalue $\lambda \in \mathbb{R}$.

The usual functional space to consider the solutions of these problems is $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, but there is another wider concept of solutions that need not to be $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Set $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and let us multiply the equation $-\Delta u = \lambda u$ by a function $v \in \mathcal{C}_0^\infty(\Omega)$,

$$-\Delta u(x)v(x) = \lambda u(x)v(x) \quad \forall x \in \Omega$$

Define $F(x) = v(x)\nabla u(x)$ for all $x \in \Omega$ and note that

$$\text{div}F(x) = \nabla v(x)\nabla u(x) + v(x)\Delta u(x) \quad \forall x \in \Omega$$

and

$$F(x) = 0 \quad \forall x \in \partial\Omega$$

By the divergence theorem,

$$\int_\Omega \text{div}F(x)\ dx = \int_{\partial\Omega} \langle F(x), \eta(x)\rangle\ dx$$

where η is the outward pointing unit normal field of the boundary. This tells us that

$$\int_\Omega (\nabla v(x)\nabla u(x) + v(x)\Delta u(x))\ dx = 0$$

or equivalently,

$$\int_\Omega \nabla v(x)\nabla u(x)\ dx = -\int_\Omega v(x)\Delta u(x)\ dx$$

Integrating over Ω the equation $-\Delta uv = \lambda uv$ and using the previous equality,

$$\int_\Omega \nabla v(x)\nabla u(x)\ dx = \lambda \int_\Omega u(x)v(x)\ dx \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$$

and by density,

$$\int_\Omega \nabla v(x)\nabla u(x)\ dx = \lambda \int_\Omega u(x)v(x)\ dx \quad \forall v \in H_0^1(\Omega)$$

This computation leads to meake the following definition.

Definition 8 (Weak solution to (P_λ))**.** *A **weak solution** to (P_λ) is a function $u \in H_0^1(\Omega)$ such that*

$$\int_\Omega \nabla v(x) \nabla u(x)\ dx = \lambda \int_\Omega u(x) v(x)\ dx \quad \forall v \in H_0^1(\Omega)$$

Obviously, every classical solution to (P_λ) is a weak solution to (P_λ). Although it is not proved here, also every weak solution to (P_λ) is a classical solution to (P_λ). The following theorem is a classical result concerning the eigenvalues of (P_λ).

Theorem 15. *If $\Omega \subset \mathbb{R}^N$ is a non-empty domain with smooth boundary, then there are infinite eigenvalues associated to the problem (P_λ) which form a increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers with the property $\lim \lambda_n = \infty$. Moreover, $X_\lambda = \{u \in H_0^1(\Omega) :$ u is solution to $(P_\lambda)\}$ is a finite-dimensional vector space with $\dim X_{\lambda_1} = 1$, $X_{\lambda_1} \oplus \cdots \oplus X_{\lambda_n} \oplus \cdots = H_0^1(\Omega)$ and if $\lambda, \mu \in \mathbb{R}$ are two different eigenvalues associated to (P_λ) then $\langle u_\lambda, u_\mu \rangle_{H_0^1(\Omega)} = 0$, where u_λ and u_μ are eigenfunctions related to the eigenvalues λ and μ, respectively.*

We will omit the proof of this theorem. Instead, we just focus on the proof of the fact that all eigenvalues are positive and $\langle u_\lambda, u_\mu \rangle_{H_0^1(\Omega)} = 0$ when $\lambda \neq \mu$ are two eigenvalues associated to (P_λ):

If $\lambda \in \mathbb{R}$ is an eigenvalue associated to (P_λ) and u_λ an eigenfunction related to λ, then

$$\int_\Omega \nabla u_\lambda(x) \nabla v(x)\ dx = \lambda \int_\Omega u_\lambda(x) v(x)\ dx \quad \forall v \in H_0^1(\Omega)$$

so if we take $v = u_\lambda$,

$$\int_\Omega |\nabla u_\lambda(x)|^2\ dx = \lambda \int_\Omega |u_\lambda(x)|^2\ dx$$

and from this it follows that $\lambda > 0$.

If $\lambda, \mu \in \mathbb{R}$ are two different eigenvalues associated to (P_λ) and u_λ and u_μ are eigenfunctions related to the eigenvalues λ and μ, respectively, then

$$\int_\Omega \nabla u_\lambda(x) \nabla v(x)\ dx = \lambda \int_\Omega u_\lambda(x) v(x)\ dx \quad \forall v \in H_0^1(\Omega)$$

$$\int_\Omega \nabla u_\mu(x) \nabla v(x)\ dx = \mu \int_\Omega u_\mu(x) v(x)\ dx \quad \forall v \in H_0^1(\Omega)$$

Replacing $v = u_\mu$ if the first equation and $v = u_\lambda$ in the second equation and subtracting,

$$0 = (\lambda - \mu) \int_\Omega u_\lambda(x) u_\mu(x)\ dx \overset{\lambda \neq \mu}{\Longrightarrow} \int_\Omega u_\lambda(x) u_\mu(x)\ dx = 0$$

but this also implies that

$$\int_\Omega \nabla u_\lambda(x) \nabla u_\mu(x)\ dx = 0$$

so one concludes

$$\langle u_\lambda, u_\mu \rangle_{H_0^1(\Omega)} = \langle u_\lambda, u_\mu \rangle_{L^2(\Omega)} + \langle \nabla u_\lambda, \nabla u_\mu \rangle_{L^2(\Omega)} = 0$$

It can be proved that there is no other eigenvalues than the found on this theorem. Besides, denoting $\varphi_k \in H_0^1(\Omega)$ a eigenfunction associated to the eigenvalue λ_k, there exist a variational characterization of the eigenvalues which is stated below:

$$\lambda_1 = \min \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}$$

$$\lambda_n = \min \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in W_n(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}$$

where

$$W_n(\Omega) = \left\{ u \in H_0^1(\Omega) : \int_\Omega \nabla u \nabla \varphi_k \, dx = 0, k = 1, ..., n-1 \right\}$$

In addition, the eigenfunction φ_1 associated to the first eigenvalue λ_1 satisfies that $\varphi_1 > 0$ (and is the unique eigenvalue with this property) and the space X_{λ_1} verifies that $\dim X_{\lambda_1} = 1$, therefore $X_{\lambda_1} = \langle \varphi_1 \rangle$ (the vector space generated by φ_1).

For example, take $N = 1$ and $I = (0, \pi)$, hence the considered problem turns to study the eigenvalues and its corresponding eigenfunctions of

$$(Q) \quad \begin{cases} -u''(x) = \lambda u(x) \text{ si } x \in I \\ u(0) = u(\pi) = 0 \end{cases}$$

The equation $u''(x) + \lambda u(x) = 0$ is a constant-coefficient linear differential equation of second order and its characteristic polynomial is $p(t) = t^2 + \lambda$ whose roots are $\pm\sqrt{-\lambda}$, so we have to distinguish three cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. It is not hard to see that if $\lambda \leq 0$, the only solution to (Q) is $u = 0$. However, if $\lambda > 0$ the solutions to $-u'' = \lambda u$ of the form

$$u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \quad \forall x \in I, \forall A, B \in \mathbb{R}$$

Since $u(0)$ must be zero, we end up with $u(x) = B\sin(\sqrt{\lambda}x)$ for all $x \in I$ $(A = 0)$. Imposing $B \neq 0$, on has

$$u(\pi) = B\sin(\sqrt{\lambda}\pi) = 0 \Leftrightarrow \lambda = n^2 \quad \forall n \in \mathbb{N}$$

Note that $\lambda = 0$ is excluded!. This means that for $\lambda_n = n^2$ with $n \in \mathbb{N}$ there is non-trivial solution to (Q), that is $\lambda_n = n^2$ are the sequence of eigenvalues and from the previous computation is follows that $u_{\lambda_n} = \sin(nx)$ is the eigenfunction associated to the eigenvalue λ_n. Here, when $N = 1$ and $\Omega = I = (0, \pi)$, all the spaces X_{λ_n} has dimension one and they are all generated by the function $\sin(nx)$, respectively. More in general, for an arbitrary interval (a, b), the eigenvalues are

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} \quad \forall n \in \mathbb{N}$$

and the corresponding eigenfunctions

$$u_{\lambda_n}(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right) \quad \forall x \in (a,b)$$

The property that the eigenfunction λ_1 associated to the first eigenvalue is the only one positive on Ω is more significant than it seems. For instance, take $N = 2$ and $\Omega = (0,\pi) \times (0,\pi)$, then the function $u_*(x,y) = \sin(x)\sin(y)$ defined on Ω is positive for all $(x,y) \in \Omega$ and also

$$-\Delta u_*(x) = 2u_*(x) \quad \forall x \in \Omega$$

which yields that $\lambda_1 = 2$ is the first eigenvalue of the Dirichlet problem on Ω and $X_{\lambda_1} = \langle \sin(x)\sin(y)\rangle$. Returning back to the Poincaré inequality, the best constant to fix is nothing but the first eigenvalue! so, for the last sample we know that

$$\int_{(0,\pi)^2} |\nabla u|^2 \, dx \geq 2 \int_{(0,\pi)^2} |u|^2 \, dx \quad \forall u \in H_0^1((0,\pi)^2)$$

and 2 is the best constant one can fix in that inequality which precisely holds only for functions of $X_{\lambda_1} = X_2 = \langle \sin(x)\sin(y)\rangle$.

What happend if one study the same problem replacing λu in the differential equation with a certain function $f = f(x)$? That is,

$$(P_f) \quad \begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$. Suppose that f satisfies the following assumptions (H)

1. $f \in L^{\frac{2N}{N+2}}(\Omega)$, if $N \geq 3$.
2. $f \in L^p(\Omega)$ for every $p > 1$, if $N = 2$.
3. $f \in L^1(\Omega)$, if $N = 1$.

By a **classical solution** to (P_f) we understand a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ which obeys the equation $-\Delta u = f$ pointwise on Ω and $u|_{\partial\Omega} = 0$. By a **weak solution** to (P_f) with f verifying (H), we understand a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega)$$

Note that the first integral is well-defined owing to the fact that u and v belong to $H_0^1(\Omega)$ and the second integral is well-defined if f fulfils (H) by virtue of the dominated convergence theorem and Hölder inequality. If the reader is interested, it is recommended to take a look at chapter nine of [8] where it is proved the existance and uniqueness of weak solution to (P_f) using the Lax-Milgram theorem and, even more, the fact that every classical solution is a weak solution and the recovery of a classical solution from a weak one. The hardest issue here is the analysis of the regularity of the weak solution: it is proved that if $f \in L^2(\Omega)$, then the weak solution to (P_f) is $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$$

for some constant $c > 0$ depending only on Ω. All this can be found in [8].

Chapter 3

The variational method

The variational method is just an astonishing analytical tool that permits to obtain theorically the existence of solutions to partial differential equations in a general background, arised from the existence of critical points of suitable functionals. The references [3] and [8] are recommended for the reader in this chapter.

3.1 Introduction to Calculus of variations

Calculus of variations is a field in mathematical analysis that study the maxima and minima of functionals defined on certain functional spaces which is very related to solving differential equations, as it will be shown. A really complete summary of the historic evolution of Calculus of variations can be read in:

(·) Erwin Kreyszig: On the Calculus of Variations ant Its Major influences on the Mathematics of the First Half of Our Century. Part I, American Mathematical Monthly, vol. 101, no. 7, 674-678, 1994.

(·) Erwin Kreyszig: On the Calculus of Variations ant Its Major influences on the Mathematics of the First Half of Our Century. Part II, American Mathematical Monthly, vol. 101, no. 9, 902-908, 1994.

According to the American historian of Mathmatics M. Kline, the first significant problem in Calculus of variations was proposed and solved by I. Newton in his second book of his work *Principia*. Newton studied the shape a surface of revolution, immersed in some fluid, must have in order to offer a minimal resistance to the movement.

Today, it is considered that Calculus of variations was born with the **Brachistochrone problem**, proposed by Johann Bernoulli. The problem is to find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point $A \in \mathbb{R}^2$ to another $B \in \mathbb{R}^2$ in the least time. Newton was challenged to solve the problem in 1696 and did so the very next day. In fact, the solution was found by Leibniz, L'Hospital, Newton, and the two Bernoullis. Johann Bernoulli solved the problem using the analogous one of considering the path of light refracted by transparent layers of varying density. Actually, Johann Bernoulli had originally found an incorrect proof that the curve is a cycloid, and challenged his brother Jakob to find the required curve. When Jakob correctly did so, Johann tried to substitute the proof for his own!.

So let be $h_0 > 0$ and two points $A = (0, h_0)$ and $B = (0, 1)$. We want to find a regular curve $f : [0, 1] \to \mathbb{R}$ connecting A with B, that is $f(0) = h_0$ and $f(1) = 0$, such that solves the Brachistochrone problem. For a fixed point $x \in [0, 1]$, we know by the energy conservation that the energy of the mass sliding down the curve f at the point x must be the same at the beginning (where there is no kinetic energy, just potential energy):

$$\frac{1}{2}mv(x)^2 + mgf(x) = mgh_0$$

So, one has the speed expression

$$v(x) = \sqrt{2g(h_0 - f(x))}$$

Since the speed is nothing but the time-derivative of the position,

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} = \frac{\sqrt{1 + f'(x)^2}dx}{v} = \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g(h_0 - f(x))}}\, dx$$

so the time that the mass employs to arrive to the point B leaving from A and following the curve f is

$$t_* = \int_0^1 \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g(h_0 - f(x))}}\, dx$$

Define the space $\Theta = \{f \in \mathcal{C}^1[0, 1] : f(0) = h_0, f(1) = 0\}$ and the funtional ϕ given by

$$\phi(f, g) = \int_0^1 \frac{\sqrt{1 + g^2}}{\sqrt{2g(h_0 - f)}}\, dx$$

where $g = f'$ and $f \in \Theta$. Now, the Brachistochrone problem has been translated to finding the minimum (if exists and is unique) of ϕ on Θ, although it is not analysed here the solution is a segment of a cycloid.

Exercise 24. *Solve the problem of the Brachistochrone.*

Another classical problem of Calculus of variations is the **Catenary problem**. The catenary is the curve a hanging flexible wire or chain assumes when supported at its ends and acted upon by a uniform gravitational force. In 1669, Jungius disproved Galileo's claim that the curve of a chain hanging under gravity would be a parabola. The equation was obtained by Leibniz, Huygens, and Johann Bernoulli in 1691 in response to a challenge by Jakob Bernoulli.

This problem is more difficult than the previous one just because it contains a ligadure or constraints. The problem in terms of mathematics is to find a curve with the end-points fiexd at height one and with a fixed length $L > 1$. Consider the space

$$\Theta = \left\{ f \in \mathcal{C}^1[0,1] : f(0) = 1, f(1) = 1, \int_0^1 \sqrt{1 + f'(x)^2}\ dx = L \right\}$$

There is only one curve in Θ that adopts its natural shape, that is, that minimize the potential energy (note there is no kinetic energy in this case because the curve lies in repose), and the potential energy is nothing but

$$E_p = mgf(x)$$

where the mass m is the density (which is considered constant) multiplied by the length or the arc, so $m = \rho\sqrt{1 + f'(x)^2}dx$ and, therefore,

$$E_p = \rho g \int_0^1 f(x)\sqrt{1 + f'(x)^2}\ dx$$

In this case the functional we would like to minimize is

$$\phi(f, g) = \int_0^1 f(x)\sqrt{1 + f'(x)^2}\ dx$$

where $g = f'$ and $f \in \Theta$, but remember that now the problem has a ligadure so in order to solve it one must introduce a Lagrange multiplier.

Exercise 25. *Solve the problem of the Catenary.*

Finally, let us present another more example: *the Hamilton principle.* Let $I \subset \mathbb{R}$ be a non-trivial interval, $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field and $x : I \to \mathbb{R}^3$ the position vector of a body under the action of the field F. By the second law of Newton,

$$mx''(t) = F(x(t)) \quad \forall t \in I$$

Watch out! this is a system of three differential equations. Assume that $m = 1$ and just write

$$x'' = F(x)$$

We are going to study the bundary value problem of that equation together with $x(0) = P$ and $x(1) = Q$, for some $P, Q \in \mathbb{R}^3$, namely

$$(\star) \quad \begin{cases} x'' = F(x) \\ x(0) = P,\ x(1) = Q \end{cases}$$

Suppose that F is a conservative vector field, meaning there exist a potential $U : \mathbb{R}^3 \to R$ with $F = -\nabla U$. Consider the functional space $\mathcal{A} = \{x \in \mathcal{C}^2([0,1], \mathbb{R}^3) : x(0) = P, x(1) = Q\}$ and the *lagrangian* $L : \mathcal{A} \to \mathbb{R}$ which is defined as

$$L(x) = \frac{1}{2}\|\dot{x}\|^2 - U(x) \quad \forall x \in \mathcal{A}$$

and finally set the functional $\Phi_* : \mathcal{A} \to \mathbb{R}$ given by

$$\Phi_*(x) = \int_0^1 L(x)\ dt = \int_0^1 \left(\frac{1}{2}\|\dot{x}\|^2 - U(x)\right)\ dt$$

Hamilton's Principle: the solutions to ($\star$) are the stationary points of Φ_* (with the assumption that F is a conservative vector field).

Before proving this, let us define what we understand by a stationary point of a general functional Φ. Consider the space $\mathcal{B} = \{h \in \mathcal{C}^2([0,1],\mathbb{R}^3) : h(0) = 0, h(1) = 0\}$ (this is sometimes called the variations space). Given $x \in \mathcal{A}$ and $h \in \mathcal{B}$, the directional derivative of Φ at x along h is

$$\Phi'(x)(h) = \frac{d}{ds}\bigg]_{s=0} \Phi(x + sh)$$

We say that $x \in \mathcal{A}$ is a *stationary (or critical) point* of Φ when $\Phi'(x)(h) = 0$ for all $h \in \mathcal{B}$.

It is worth noting that every minimum of Φ is a stationary point. This is a analogous result to the well-known for functions from $\mathbb{R}$ to $\mathbb{R}$: if $f : I \subset \mathbb{R} \to \mathbb{R}$ is derivable at $x_0 \in I$ and x_0 is a minimum of f on I, then $f'(x_0) = 0$. Obviously, the reciprocal is false. To prove that every minimum of Φ is a stationary point just define the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = \Phi(x + sh) \quad \forall s \in \mathbb{R}$$

So, if Φ has a minimum at x then

$$\psi(s) = \Phi(x + sh) \geq \Phi(x) = \psi(0) \quad \forall s \in \mathbb{R}$$

that is, ψ has a minimum at zero (independently of $h \in \mathcal{B}$) and consequently,

$$\psi'(0) = \frac{d}{ds}\bigg]_{s=0} \Phi(x + sh) = 0 \quad \forall h \in \mathcal{B}$$

which means that x is a stationary point of Φ. The proof of Hamilton's Principle is done next.

proof (Hamilton's Principle). In Hamilton's Principle, the functional Φ is

$$\Phi(x) = \int_0^1 \left(\frac{1}{2}\|\dot{x}\|^2 - U(x)\right)\, dt$$

then one has

$$\begin{aligned}
\Phi'_*(x)(h) &= \int_0^1 \frac{\partial}{\partial s}\left[\frac{1}{2}\|\dot{x}+s\dot{h}\|^2 - U(x+sh)\right]\, dt = \\
&= \int_0^1 \left\{(\dot{x}(t)+s\dot{h}(t))\cdot\dot{h}(t) - \nabla U(x(t)+sh(t))\cdot h(t)\right\}|_{s=0}\, dt = \\
&= \int_0^1 \left\{\dot{x}(t)\cdot\dot{h}(t) - \nabla U(x(t))\cdot h(t)\right\}\, dt = \\
&= \dot{x}(t)h(t)|_0^1 - \int_0^1 \ddot{x}(t)h(t)\, dt - \int_0^1 \nabla U(x(t))\cdot h(t)\, dt = \\
&= -\int_0^1 \left(\ddot{x}(t)+\nabla U(x(t))\right)h(t)\, dt
\end{aligned}$$

If x is a solution to $(\star)$, then $x \in \mathcal{A}$ and $x'' = F(x) = -\nabla U(x)$ and so $\ddot{x} + \nabla U(x) = 0$, which implies that $\Phi'_*(x)(h) = 0$ (for every $h \in \mathcal{B}$) and so, x is a stationary point of Φ_*.

If x is a statonary point of Φ, then $\Phi'_*(x)(h) = 0$ for all $h \in \mathcal{B}$, which implies

$$\int_0^1 \left(\ddot{x}(t)+\nabla U(x(t))\right)h(t)\, dt = 0 \quad \forall h \in \mathcal{B}$$

and by the fundamental lemma of Calculus of variations, it follows that $\ddot{x} = -\nabla U(x) = F(x)$, as wanted. $\square$

The Principle of Hamilton asserts that the solutions of $(\star)$, are the stationary points of the functional ϕ_* and vice versa. This fact can be extended to a more general background and to do it, we need to introduce the so-called *Euler-Lagrange equations*. Given two real numbers a and b with $a < b$, the real interval $I = [a, b]$ and $N \in \mathbb{N}$, the function $\mathcal{L} : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is named *lagrangian* and one can define in general the functional $\Phi : \mathcal{A} \to \mathbb{R}$ given by

$$\Phi(x) = \int_a^b \mathcal{L}(t, x, x')\, dt \quad \forall x \in \mathcal{A}$$

where $\mathcal{A} = \{x \in \mathcal{C}^2([a, b], \mathbb{R}^N) : x(a) = P, x(b) = Q\}$. The equations of Euler-Lagrange are the following system of N E.D.O.'s

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} = 0$$

In a more general situation, the Hamilton's Principle claims that stationary points of Φ are corresponded with solutions to Euler-Lagrange equations.

Exercise 26. *Prove the previous assertion (Hamilton's Principle).*

Remark 13. *In the case that $\mathcal{L}(t, x, x') = \frac{1}{2}\|x'\|^2 - U(x)$ with $F = -\nabla U$, the Euler-Lagrange equations are nothing but $x'' = F(x)$.*

The Euler-Lagrange equations have a confortable equivalent expression when the lagrangian does not depend explicitly on t. If $\mathcal{L} = \mathcal{L}(x, x')$, then we write the following

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial x}x' + \frac{\partial \mathcal{L}}{\partial x'}x'' \Rightarrow \frac{\partial \mathcal{L}}{\partial x}x' = \frac{d\mathcal{L}}{dt} - \frac{\partial \mathcal{L}}{\partial x'}x''$$

and mutliplying the Euler-Lagrange equations by x' and replacing $\frac{\partial \mathcal{L}}{\partial x}x'$ by $\frac{d\mathcal{L}}{dt} - \frac{\partial \mathcal{L}}{\partial x'}x''$

$$x'\frac{\partial \mathcal{L}}{\partial x} - x'\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'}\right) = 0 \Rightarrow -\frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\left[\mathcal{L} - x'\frac{\partial \mathcal{L}}{\partial x'}\right] = 0$$

Since $\mathcal{L} = \mathcal{L}(x, x')$, the time-partial derivative is zero and thus the Euler-Lagrange equations are equivalent to

$$\frac{d}{dt}\left[\mathcal{L} - x'\frac{\partial \mathcal{L}}{\partial x'}\right] = 0$$

that is

$$\mathcal{L} - x'\frac{\partial \mathcal{L}}{\partial x'}$$

is constant in time. This is often known as the Beltrami Principle.

Remark 14. *In the case $\mathcal{L}(t, x, x') = \mathcal{L}(x, x') = \frac{1}{2}\|x'\|^2 - U(x)$ with $F = -\nabla U$, the quantity (=total energy)*

$$\mathcal{L} - x'\frac{\partial \mathcal{L}}{\partial x'} = -\frac{1}{2}\|x'\|^2 - U(x)$$

is a time-invariant.

The aim of this introduction is to show the importance of finding stationary points of certain functionals and its relationship with differential equations. Our big goal is now minimize functionals defined on Banach spaces of arbitrary dimension, because every minimum of a functional turns out to be a stationary point (but not conversely: a statinary point can also be a maximum point or a saddle point!). We are going to focus on the particular case of Hilbert spaces.

3.2 Minimization of functionals on Hilbert spaces

As it was said, the main goal now is to minimize or maximize real-valued functionals defined on a Hilbert space X of arbitrary dimension. Obviously, a function f has a minimum at a point x_* if, and only if, $-f$ has a maximum at x_*, hence one can restrint to the case of finding minimum points.

Let us start with a finite-dimensional Banach space X. Without loss of generality, we can suppose that $X = \mathbb{R}^N$ for some $N \in \mathbb{N}$. Set $F : \mathbb{R}^N \to \mathbb{R}$ a function, what conditions can we required of F to have a minimum? Just two assumptions: *continuity* and *coercivity.* Recall that $F : \mathbb{R}^N \to \mathbb{R}$ is

$$\text{coercive} \Leftrightarrow \lim_{\|x\|\to\infty} F(x) = +\infty$$

and

$$\begin{aligned}
\text{continuous at } x \in \mathbb{R}^N \quad &\Leftrightarrow \quad [V \in \mathcal{U}_{\mathbb{R}^N}(F(x)) \Rightarrow F^{-1}(V) \in \mathcal{U}_{\mathbb{R}}(x)] \\
&\Leftrightarrow \quad \left[\forall \varepsilon > 0\ \exists \delta > 0 : y \in \mathbb{R}^N, \|y - x\| < \delta \Rightarrow |F(y) - F(x)| < \varepsilon\right] \\
&\Leftrightarrow \quad \left[\forall \{x_n\} \subset \mathbb{R}^N, x_n \to x \Rightarrow F(x_n) \to F(x)\right]
\end{aligned}$$

where $\mathcal{U}_{\mathbb{R}^N}(F(x))$ denotes the set of neighbourhoods of the point $F(x)$ in $\mathbb{R}^N$ and similarly, $\mathcal{U}_{\mathbb{R}}(x)$ denotes the set of neighbourhoods of the point x in $\mathbb{R}$.

Our first result claims that continuity and coercivity are enough for F to attain its minimum. Intuitively, these are two good conditions in order to find (global) minimums of real-valued functions defined on $\mathbb{R}^N$: continuity of F provides that F attains a global minimum on $\overline{B}(0, \rho)$ for some $\rho > 0$ and coercivity implies that F is bounded from below outside the ball $\overline{B}(0, \rho)$.

Lemma 1. *If $F : \mathbb{R}^N \to \mathbb{R}$ is continuous and coercive, then F attains its minimum.*

Proof. Set $\alpha = \inf F \in [-\infty, \infty)$. By characterization of infimum, there is a sequence $\{x_n\} \subset \mathbb{R}^N$ such that $F(x_n) \to \alpha$. By coercivity, if $\|x_n\| \to \infty$, then $F(x_n) \to +\infty$ and so $\alpha = \infty$. This shows that $\{x_n\}$ cannot diverge and therefore $\{x_n\}$ is bounded. By Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{\sigma(n)}\}$ of $\{x_n\}$ that is convergent to a point $x_0 \in \mathbb{R}^N$. By continuity of F, on has $F(x_{\sigma(n)}) \to F(x_0)$, but since $F(x_n) \to \alpha$ and $\{x_{\sigma(n)}\}$ is a subsequence of $\{x_n\}$, it forces that $F(x_0) = \alpha$. Finally, since F takes real values, it follows that $\alpha \in \mathbb{R}$ (or in other words $\alpha > -\infty$, which basically means that F is bounded from below) and the infimum α is attained at the point x_0, so it is in fact the global minimun of F. $\square$

This first result is nice, but it can be improved a lot! To see how, we must look further into the proof just done:

Firstly, it is not needed the continuity of F at all. We can still make the same proof with a weaker assumption concerning the continuity of F. Note that in one step of the proof, we wrote $x_{\sigma(n)} \to x_0 \overset{\text{F continuous}}{\Longrightarrow} F(x_{\sigma(n)}) \to F(x_0) \overset{F(x_{\sigma(n)})\to\alpha}{\Longrightarrow} \alpha = F(x_0)$ and this can be replaced by

$$x_{\sigma(n)} \to x_0 \Longrightarrow F(x_0) \leq \liminf F(x_{\sigma(n)}) \overset{F(x_{\sigma(n)})\to\alpha}{=} \alpha \Longrightarrow F(x_0) \leq \alpha$$

Since $\alpha = \inf F$ there is nothing left to say that $F(x_0) = \alpha$ and this would complete the proof. In other words, it is enough to replace the continuity by the condiction that if $x_{\sigma(n)} \to x_0$, then $F(x_0) \leq \liminf F(x_{\sigma(n)})$ and this is named the *lower semicontinuity.*

Definition 9 (Lower semicontinuity). *Given a metric space X, a function $f : X \to \mathbb{R}$ is* ***lower semicontinuous at*** *$x \in X$ if for every sequence $\{x_n\} \subset X$ with $x_n \to x$, then $f(x) \leq \liminf f(x_n)$. If f is lower semicontinuous at every point $x \in X$, then f is said to be* ***lower semicontinuous****.*

Secondly, as we are looking for minimum points of F it does not matter that F takes the value $+\infty$, provided that F is not constantly $+\infty$. Overall, we can state a new and improved version of lemma 1.

Corollary 5. *If $F : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, coercive and $F \not\equiv +\infty$, then F attains its minimum.*

This new version is better than the first one, but we still can enhance it by introducing the concept of *weakly lower semicontinuity.* Before that, let us recall what do we understand by *weak convergence.* In general, given a Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$, we say that a sequence $\{x_n\} \subset H$ is convergent to a limit $x \in H$ when $\|x_n - x\| \to 0$ as $n \to \infty$, where $\| \cdot \|$ is nothing but $\sqrt{\langle \cdot, \cdot \rangle}$. There is another wider concept of convergence here that is introduced next.

Definition 10 (Weak convergence). *Given a Hilbert space H, a sequence $\{x_n\} \subset H$* ***converges weakly*** *to $x \in H$ if $\langle x_n, y \rangle \to \langle x, y \rangle$ for all $y \in H$ and this is denoted by $x_n \rightharpoonup x$.*

Definition 11 (Weak continuity). *Given a Hilbert space H, a function f is* ***weakly continuous*** *if*

$$H \supset x_n \rightharpoonup x \Rightarrow f(x_n) \to f(x)$$

Definition 12 (Weak lower semicontinuity). *Given a Hilbert space H, a function f is* ***weakly lower semicontinuous*** *if*

$$H \supset x_n \rightharpoonup x \Rightarrow f(x) \leq \liminf_{n \to \infty} f(x_n)$$

Remark 15. *Obviously, every convergent sequence of a Hilbert space is also weakly convergent (to the same limit) because the scalar product is a continuous map from $H \times H$ to $\mathbb{R}$. The converse is false as it is shown in the following example.*

Example 8. *Take $H = L^2(0, \pi)$ with the usual scalar product $\langle f, g \rangle = \langle f, g \rangle_{L^2(0,\pi)}$ for each $f, g \in H$. Basic Fourier analysis asserts that*

$$f = \sum_{n=1}^{\infty} a_n \sin(n\cdot) \qquad \forall f \in H \tag{3.1}$$

where the coefficient a_n are given by the formula

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \qquad \forall n \in \mathbb{N}$$

and the equality in 3.1 means $L^2(0, \pi)$-convergence, that is

$$\| \sum_{n=1}^{m} a_n \sin(n\cdot) - f \|_{L^2(0,\pi)} \longrightarrow 0 \quad (m \to +\infty)$$

Actually, by Parseval identity, for each $f \in H$

$$\|f\|_{L^2(0,\pi)} = \sum_{n=1}^{\infty} a_n^2 \|\sin(n\cdot)\|_{L^2(0,\pi)} = \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2$$

which, among other things, implies that $a_n \to 0$.

Consider the sequence in H defined by $y_n(x) = \sin(nx)$ for all $x \in (0,\pi)$ and for all $n \in \mathbb{N}$. On the one hand, this sequence is weakly convergent to zero! Since $a_n \to 0$ and $a_n = 2/\pi\langle f, y_n\rangle$ for all $f \in H$, it follows that $\langle f, y_n\rangle \to 0 = \langle f, 0\rangle$, which is precisely the definition of weak convergence. On the other hand, this sequence is not convergent, otherwise one would have that $y_n \to 0$ (by remark 15) and, also $\|y_n\| \to 0$ but

$$\|y_n\|^2 = \int_0^\pi \sin(nx)^2 \; dx = \frac{\pi}{2} \quad \forall n \in \mathbb{N}$$

This example also shows that the norm map $\|\cdot\|$, which is continuous on $H \times H$, is not weakly continuous, i.e.

$$x_n \rightharpoonup x \not\Rightarrow \|x_n\| \to \|x\|$$

Although, it can be proved that the norm $\|\cdot\|$ on every Hilbert space is one the esasiest examples of weakly lower semicontinuous function. In other words, what weak convergence needs to be strong convergence is the convergence of the norm.

Proposition 20. *Let H be a Hilbert space, then $\|\cdot\| : H \times H \to \mathbb{R}$ is weakly lower semicontinuous.*

Proof. Let $\{x_n\} \subset H$ be a weakly convergent sequence to $x \in H$. In case that $x = 0$, there is nothing left to prove, in other case it is clear that

$$\langle x_n, \frac{x}{\|x\|}\rangle \to \langle x, \frac{x}{\|x\|}\rangle = \|x\|$$

By Cauchy-Schwarz inequality

$$|\langle x_n, \frac{x}{\|x\|}\rangle| \leq \|x_n\|$$

so taking $\liminf$ in both sides of the inequality, one has

$$\|x\| = \liminf |\langle x_n, \frac{x}{\|x\|}\rangle| \leq \liminf \|x_n\|$$

and this complete the proof. □

Let us see now two useful properties of the weak convergence.

Proposition 21. *Every weakly convergent sequence of a Hilbert space is bounded.*

Proof. It is a consequence of Uniform boundedness principle. □

Proposition 22. *Let H be a Hilbert space and $\{y_n\} \subset H$ a sequence, then*

$$y_n \to y \Leftrightarrow \begin{cases} y_n \rightharpoonup y \\ \|y_n\| \to \|y\| \end{cases}$$

Proof.
$\Rightarrow$) Trivial.
$\Leftarrow$)

$$\|y_n - y\|^2 = \langle y_n - y, y_n - y\rangle = \|y\|^2 + \|y_n\|^2 - 2\langle y_n, y\rangle \to \|y\|^2 + \|y\|^2 - 2\langle y, y\rangle = 0$$

□

The next result allows us to generalize lemma 1. It could be sawn as a Bolzano-Weiertrass theorem for Hilbert spaces (of arbitrary dimension) taking into account the weak convergence instead.

Proposition 23. *Every bounded sequence of a Hilbert space has a weakly convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence in a Hilbert space H and define $H_0 = \overline{\text{span}(x_n : n \in \mathbb{N})}$, then H_0 is separable as the set of all finite linear combinations of points in $\{x_n\}$ with rational coefficients is a countable and dense subset of H_0. For each $n \in \mathbb{N}$, consider $f_n(x) = \langle x_n, x\rangle$ for all $x \in H_0$, then f_n is clearly linear and also bounded (with the norm of H_0^*) by virtue of Cauchy-Schwarz inequality and the boundness of $\{x_n\}$. Helley's theorem[1] ensures that there is a subsequence $\{f_{\sigma(n)}\}$ that converges weakly in H_0^* to a function $f_0 \in H_0^*$, that is $f_{\sigma(n)}(x) \to f_0(x)$ for all $x \in H_0$. On the other hand, by Riesz-Representation theorem there is a unique element $x_* \in H_0$ such that $f_0(x) = \langle x, x_*\rangle$ for each $x \in H_0$, and

$$\lim\langle x, x_{\sigma(n)}\rangle = \lim f_{\sigma(n)}(x) = f_0(x) = \langle x, x_*\rangle \quad \forall x \in H_0 \qquad (\star)$$

Finally, if $x \in H$ and P_{H_0} is the orthogonal projection of H onto H_0, then

$$\begin{aligned} \langle x, x_{\sigma(n)}\rangle &\overset{x_{\sigma(n)} \in H_0}{=} \langle x, P_{H_0}(x_{\sigma(n)})\rangle \overset{(**)}{=} \\ &= \langle \underbrace{P_{H_0}(x)}_{\in H_0}, x_{\sigma(n)}\rangle \overset{\star}{\to} \langle P_{H_0}(x), x_*\rangle \overset{(**)}{=} \\ &= \langle x, P_{H_0}(x_*)\rangle \overset{x_* \in H_0}{=} \langle x, x_*\rangle \end{aligned}$$

and this exactly means that $x_{\sigma(n)} \rightharpoonup x_*$, where $(**)$ we are making the next reasoning

$$\begin{aligned} \langle P_{H_0}(x), y\rangle &= \langle P_{H_0}(x), P_{H_0}(y) + (y - P_{H_0}(y))\rangle = \\ &= \langle P_{H_0}(x), P_{H_0}(y)\rangle + \langle \underbrace{P_{H_0}(x)}_{\in H_0}, \underbrace{y - P_{H_0}(y)}_{\in H_0^\perp}\rangle = \langle P_{H_0}(x), P_{H_0}(y)\rangle \end{aligned}$$

[1] Helley's theorem: if X is a normed linear space and separable, then every bounded sequence of continuous linear functionals in X^* has a subsequence that weakly converges in X^*.

Similarly,

$$\langle x, P_{H_0}(y)\rangle = \langle P_{H_0}(x), P_{H_0}(y)\rangle$$

so, $\langle P_{H_0}(x), y\rangle = \langle x, P_{H_0}(y)\rangle$. □

It is also worthy to mention that in $\mathbb{R}^N$ for every $N \in \mathbb{N}$, the notion of weak convergence is equivalent to strong convergence, i.e. if $\{x_n\} \subset \mathbb{R}^N$ satisfies $x_n \rightharpoonup x$ for some $x \in \mathbb{R}^N$ then it also verifies $x_n \to x$. This easily follows from the definition of weak convergence applied to the elements of the basis $B = \{e_1, ..., e_N\}$ where e_k is the vector with all the components zero but a one in the k-th position.

At this point, we can state a new more version of lemma 1.

Corollary 6. *If H is a Hilbert space and $F : H \to \mathbb{R}\cup\{+\infty\}$ is weakly lower semicontinuous, coercive and $F \not\equiv +\infty$, then F attains its minimum.*

Remark 16. *It remains to say that coercivity is only needed to get the boundedness of the minimizing sequence, so the coercivity is not needed if one knows that there is a bounded minimizing sequence in advance.*

Corollary 6 has many applicactions in Differential Equations. Here, we present two examples of it.

3.2.1 Abstract example

Let us consider the problem

$$(P)\ \begin{cases} -\Delta u + |u|^{p-1}u = f(x) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain and

$(H1)$ $2 < p + 1 < 2^* = \frac{2N}{N-2}$ if $N > 2$

$(H2)$ $f \in L^2(\Omega)$

A function $u \in H_0^1(\Omega)$ is a *weak solution* to (P) if

$$\int_\Omega \nabla u \nabla v\ dx + \int_\Omega |u|^{p-1}uv\ dx - \int_\Omega fv\ dx = 0 \qquad \forall v \in H_0^1(\Omega)$$

If one defines the functional I on $H_0^1(\Omega)$ by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}\ dx - \int_\Omega fu\ dx$$

then the critial points of I turn out to be weak solutions of (P) since

$$I'(u)(v) = \int_\Omega \nabla u \nabla v\ dx + \int_\Omega |u|^{p-1}uv\ dx - \int_\Omega fv\ dx$$

Obviously, our functional I is well defined. The first integral makes sense as $u \in H_0^1(\Omega)$ and so $\nabla u \in L^2(\Omega)$, the second integral makes sense in view of $(H1)$ and Sobolev embedding theorem and the third integral makes sense just because $u \in L^2(\Omega)$ and $(H2)$. As it was said in the Introduction of this chapter, every minimum points of I is in fact a critical point. Can we find minimizers of I? Well, yes! We have already stated a result that can help us to do it, namely corollary 6.

Clearly, $H = H_0^1(\Omega)$ is a Hilbert space and I is not constantly $+\infty$. In order to show that I admits a minimizer (which leads to a weak solution of (P)), all we have to do is check the weak lower semicontinuity and the coercivity of I.

Weakly lower semicontinuity: given a sequence $\{u_n\}$ with $u_n \rightharpoonup u$, we write the following

$u_n \rightharpoonup u$ in $H_0^1(\Omega)$ implies $u_n \rightharpoonup u$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$, and the weak lower semicontinuity of the norm $\|\nabla u\|_{L^2(\Omega)}$ gives,

$$\liminf \int_\Omega |\nabla u_n|^2 \, dx = \liminf \|\nabla u_n\|^2_{L^2(\Omega)} \geq \|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 \, dx$$

by the Rellich-Kondrachov theorem,

$$\int_\Omega |u_n|^{p+1} \, dx \to \int_\Omega |u|^{p+1} \, dx \Rightarrow \liminf \int_\Omega |u_n|^{p+1} \, dx = \int_\Omega |u|^{p+1} \, dx$$

by the definition of weak convergence of $\{u_n\}$ in $L^2(\Omega)$,

$$\int_\Omega f u_n \, dx = \langle f, u_n \rangle \to \langle f, u \rangle = \int_\Omega f u \, dx \Rightarrow \liminf \int_\Omega f u_n \, dx = \int_\Omega f u \, dx$$

These arguments are OK, but we can get rid of the hypothesis $(H1)$ using a slightly subtle reasoning. Note that hypothesis $(H1)$ is needed to apply Rellich-Kondrachov theorem and with the next new argument that theorem is not needed at all. This new argument uses two results that we now state without a proof. If the reader is interested, the proofs can be found in [8].

Lemma 2 (Fatou). *If $f_n : \Omega \to [0, \infty]$ is a sequence of positive and measurable functions on $\Omega \subset \mathbb{R}^N$ such that $\{f_n\}$ converges pointwise to a function f for almost every point in Ω, then*

$$\liminf \int_\Omega f_n(x) \, dx \geq \int_\Omega f(x) \, dx$$

Lemma 3. *If $p \in [1, \infty]$ and $f_n : \Omega \to \mathbb{R}$ is a sequence of measurable functions on $\Omega \subset \mathbb{R}^N$ such that $\{f_n\}$ converges in $L^p(\Omega)$ to a function f, then there exists a subsequence $\{f_{\sigma(n)}\}$ of $\{f_n\}$ that converges pointwise to f for almost every point in Ω and there exists a function $g \in L^p(\Omega)$ with $|f_{\sigma(n)}(x)| \leq g(x)$ a.e. on Ω.*

Clearly, $u_n \to u$ in $L^2(\Omega)$ by Rellich-Kondrachov theorem, hence lemma 3 guarantees the existence of a subsequence $\{u_{\sigma(n)}\}$ converging pointwise to u a.e. on Ω. Also, $|u_{\sigma(n)}|^{p+1}$ converges pointwise to $|u|^{p+1}$, thus lemma 2 yields that

$$\liminf \int_\Omega |u_{\sigma(n)}(x)|^{p+1}\, dx \geq \int_\Omega |u(x)|^{p+1}\, dx$$

This is almost what we want. In order to get the above inequality for the complete sequence $\{u_n\}$ and not up to a subsequence, set $L = \liminf \int_\Omega |u_n(x)|^{p+1}\, dx$ and $\tau = \tau(n)$ such that $\int_\Omega |u_{\tau(n)}|^{p+1}\, dx \to L$ and repeat the previous argument to $\{u_{\tau(n)}\}$.

Coercivity: this follows from the Poincaré's inequality and Hölder's inequality. Again, it is also appropiate now to use the equivalent norm $\|u\| = \|\nabla u\|_{L^2(\Omega)}$ in $H_0^1(\Omega)$,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx + \frac{1}{p+1}\int_\Omega |u(x)|^{p+1}\, dx - \int_\Omega f(x)u(x)\, dx \geq \\
&\geq \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \int_\Omega f(x)u(x)\, dx \geq \\
&\geq \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \left(\int_\Omega f(x)^2\, dx\right)^{1/2}\left(\int_\Omega u(x)^2\, dx\right)^{1/2} \geq \\
&= \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \left(\int_\Omega f(x)^2\, dx\right)^{1/2}\left(\int_\Omega u(x)^2\, dx\right)^{1/2} \geq \\
&\geq \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \left(\int_\Omega f(x)^2\, dx\right)^{1/2}\left(C\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2} = \\
&= \tfrac{1}{2}\|u\|^2 - C'\|u\| \to \infty \quad (\text{as } \|u\| \to \infty)
\end{aligned}$$

Application of corollary 6 provides a minimizer of I and, consequently, a weak solution to (P). This completes the existence of the problem (P). What about the unicity of solution? this is left as an exercise for the reader.

Exercise 27. *Show that problem (P) has a unique solution.*

3.2.2 Nonlinear simple pendulum

Let us discuss another example: the nonlinear pendulum equation with a external force together with periodic boundary conditions. If $u = u(t)$ represents the time-variant angle that form the rope where the mass is held with respect to the a fixed body, just like the image below

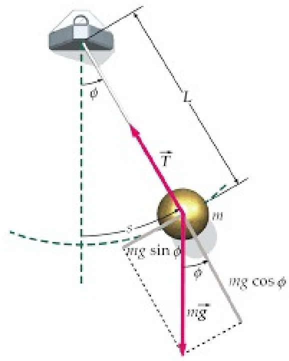

Figure 3.1: Simple pendulum

then the problem to solve, when there is a external T-periodic force $h \in \mathcal{C}^1(\mathbb{R})$ acting on the mass, is

$$(Q) \quad \begin{cases} u'' + \sin u = h \text{ on } \mathbb{R} \\ u(t+T) = u(t) \text{ for all } t \in \mathbb{R} \end{cases}$$

Recall that when the oscillations are small enough, one can approximate the problem with another lineal one since $\sin u \approx u$. An easy and interesting result is that if the mean value of h (see definition below) does not belong to $[-1, 1]$, then there are no T-periodic solutions to (Q).

Definition 13 (Mean of a function)**.** *Given an non-trivial interval $I = [a, b]$, the **mean value** of a function $h \in L^1(I)$ is the average value of the function over its domain, that is*

$$\overline{h} = \frac{1}{b-a} \int_a^b h(t)\ dt$$

Proposition 24. *If the problem (Q) admits solution, then $|\overline{h}| \leq 1$.*

Proof. Suppose that u is a solution of (Q) so, in particular is T-periodic as well as its derivative u' [2]. Integrating the equation $u'' + \sin u = h$ on $[0, T]$, one gets the desired result

$$\underbrace{\int_0^T u''(t)\ dt}_{u'(T)-u'(0)=0} + \int_0^T \sin u(t)\ dt = \int_0^T h(t)\ dt;$$

$$\left| \int_0^T h(t)\ dt \right| = \left| \int_0^T \sin u(t)\ dt \right| \leq \int_0^T |\sin u(t)|\ dt \leq T \Rightarrow |\overline{h}| \leq 1$$

□

[2]Be careful! If u is T-periodic then u' is also T-periodic, but the converse is false: in general the T-periodicity of u' does not imply the T-periodicity of u and it is not difficult to find counterexamples.

Let us start introducing the variational formulation of (Q). Firstly, consider the functional space where our problem is well-posed,

$$H^1_T(\mathbb{R}) = \{u \in H^1_{loc} : u(x) = u(x+T), x \in \mathbb{R}\}$$

Secondly, consider the $\mathcal{C}^1$-functional $I : H^1_T(\mathbb{R}) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_0^T |u'(t)|^2 \, dt + \int_0^T \cos(u(t)) \, dt - \int_0^T h(t)u(t) \, dt \quad \forall u \in H^1_T(\mathbb{R})$$

Clearly, critical points of J are T-periodic weak solutions to (Q). All we have to do is prove the existence of critical points of J, for examples, minimizers of J. Unfourtunately, corollary 6 cannot be applied in this case because J is not coercive in general! This is easy to check, since constant functions (which of course belong to $H^1_T(\mathbb{R})$) makes I tends to $-\infty$,

$$J(\kappa) = \int_0^T \cos(\kappa) \, dt - \int_0^T h(t)\kappa \, dt = T\cos(\kappa) - T\kappa\overline{h}$$

so,

1. if $\overline{h} > 0$, $u_n(t) \equiv \kappa_n \to +\infty \Rightarrow I(u_n) \to -\infty$
2. if $\overline{h} < 0$, $u_n(t) \equiv \kappa_n \to -\infty \Rightarrow I(u_n) \to -\infty$

The question is: what happens when $\overline{h} = 0$?

Theorem 16. *If $\overline{h} = 0$, then J attains its minimum.*

Remark 17. *If the reader remembers the proof of corollary 6, then it must be clear that coercivity is only required to get the boundness of a minimizing sequence for the functional, so it seems that now, we will have to manage to get it from other arguments.*

Before doing the proof of 16, is better to recall a really useful inequality called Poincaré-Wirtinger inequality.

Lemma 4 (Poincaré-Wirtinger inequality)**.** *If $f \in H^1(\mathbb{R})$ is $(b-a)-$ periodic $(f(a) = f(b))$ and $\overline{f} = 0$, then*

$$\int_a^b f'(t)^2 \, dt \geq \frac{4\pi^2}{(b-a)^2}\int_a^b f(t)^2 \, dt$$

Exercise 28. *Prove the Poincaré-Wirtinger inequality.*

Proof of theorem 16. Consider $\beta := \inf_{H^1_T(\mathbb{R})} J \in [-\infty, \infty)$ and $\{u_n\} \subset H^1_T(\mathbb{R})$ such that $J(u_n) \to \beta$ and set

$$\overline{u_n} = \frac{1}{T}\int_0^T u_n(t) \, dt \quad \forall n \in \mathbb{N}$$

and $\tilde{u_n} = u_n - \overline{u_n}$ for all $n \in \mathbb{N}$. Note that

$$\int_0^T \tilde{u_n}(t) \, dt = \int_0^T u_n(t) - \overline{u_n} \, dt = T\overline{u_n} - T\overline{u_n} = 0$$

this means that we have wroten $\{u_n\}$ as the sum of two terms ($\tilde{u_n}$ and $\overline{u_n}$) where the mean of $\tilde{u_n}$ is zero. Rename $\tilde{v_n} := \tilde{u_n}$ and define $v_n = \tilde{v_n} + \overline{v_n}$ for each $n \in \mathbb{N}$. Without loss of generality, one can assume that $\overline{v_n} \in [0, 2\pi)$ and $u_n - v_n = \overline{u_n} - \overline{v_n}$ for every $n \in \mathbb{N}$. Note that

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int_0^T v_n'(t)^2\, dt + \int_0^T \cos(v_n(t))\, dt - \int_0^T h(t)v_n(t)\, dt = \\ &= \frac{1}{2}\int_0^T \tilde{v_n}'(t)^2\, dt + \int_0^T \cos(u_n(t))\, dt - \int_0^T h(t)\tilde{v_n}(t)\, dt = \\ &= \frac{1}{2}\int_0^T u_n'(t)^2\, dt + \int_0^T \cos(u_n(t))\, dt - \int_0^T h(t)u_n(t)\, dt = I(u_n) \end{aligned}$$

Since $J(u_n) \to \beta$, the previous computation implies $J(v_n) \to \beta$, i.e. $\{v_n\}$ is also a minimizing sequence. One would like to say that $\{v_n\}$ is bounded (and "copy" the proof of corollary 6), but $v_n = \tilde{v_n} + \overline{v_n}$ and $\overline{v_n} \in [0, 2\pi)$, so in order to $\{v_n\}$ be bounded one needs the boundness of $\{\tilde{v_n}\}$. Is $\{\tilde{v_n}\}$ bounded? Yes, in view of Poincaré-Wirtinger inequality. We make the following computation where we use the fact that $|\cos| \leq 1$, Hölder inequality and finally Poincaré-Wirtinger inequality.

$$\begin{aligned} I(v_n) &\geq \frac{1}{2}\int_0^T \tilde{v_n}'(t)^2\, dt - T - \left(\int_0^T h(t)^2\, dt\right)^{1/2} \left(\int_0^T \tilde{v_n}(t)^2\, dt\right)^{1/2} \\ &\geq \frac{1}{2}\int_0^T \tilde{v_n}'(t)^2\, dt - T - \left(\int_0^T h(t)^2\, dt\right)^{1/2} \frac{T}{4\pi^2} \left(\int_0^T \tilde{v_n}'(t)^2\, dt\right)^{1/2} = \\ &= \frac{1}{2}\int_0^T \tilde{v_n}'(t)^2\, dt - T - C\left(\int_0^T \tilde{v_n}'(t)^2\, dt\right)^{1/2} \to \beta \end{aligned}$$

This implies that

$$\int_0^T \tilde{v_n}'(t)^2\, dt \text{ and } \int_0^T \tilde{v_n}(t)^2\, dt$$

are bounded or, in other words, $\{\tilde{v_n}\}$ is bounded in $H^1_T(\mathbb{R})$, therefore $\{v_n\}$ is bounded in $H^1_T(\mathbb{R})$. By proposition 23, there exists $v \in H^1_T(\mathbb{R})$ and a subsequence $\{v_{\sigma(n)}\}$ of $\{v_n\}$ such that $v_{\sigma(n)} \rightharpoonup v$ in $H^1_T(\mathbb{R})$. At this point, if we establish the weakly lower semicontinuity of J, we are done!.

$v_{\sigma(n)} \rightharpoonup v$ in $H^1[0,T]$ implies $v_{\sigma(n)} \rightharpoonup v$ and $v'_{\sigma(n)} \rightharpoonup \nabla v'$ in $L^2[0,T]$, and the weak lower semicontinuity of the norm $\|v'\|_{L^2[0,T]}$ gives,

$$\liminf \int_\Omega |v'_{\sigma(n)}|^2\, dt = \liminf \|v'_{\sigma(n)}\|^2_{L^2(0,T)} \geq \|v'\|^2_{L^2(0,T)} = \int_\Omega |v'|^2\, dt$$

by the Rellich-Kondrachov theorem, the space $H^1[0,T]$ is compactly embedded in the space of continuous functions $\mathcal{C}[0,T]$. Consequently, $v_{\sigma(n)} \to v$ in $\mathcal{C}[0,T]$, that is $\{v_{\sigma(n)}\}$ uniformly converges to v in $[0,T]$ and, as a consequence, also $\{v_{\sigma(n)}\}$ pointwise converges to v in $[0,T]$. By an obvious application of the dominated convergence theorem,

$$\int_0^T \cos(v_{\sigma(n)})\ dt \to \int_0^T \cos(v)\ dt$$

by the definition of weak convergence of $\{v_{\sigma(n)}\}$ in $L^2[0,T]$,

$$\int_0^T hv_{\sigma(n)}\ dt = \langle h, v_{\sigma(n)}\rangle \to \langle h, v\rangle = \int_0^T hv\ dt$$

This complete the weak lower semicontinuity of J. The only thing left to do is

$$\liminf J(v_{\sigma(n)}) \geq J(v)$$

Again this argument was done previously: since $J(v_n) \to \beta$ and $\{v_{\sigma(n)}\}$ is a subsequence of $\{v_n\}$, also $J(v_{\sigma(n)}) \to \beta$, then

$$\beta = \liminf J(v_{\sigma(n)}) \geq J(v) \Rightarrow J(v) = \beta$$

Namely, J attains its minimum at v. □

Some remarks are now worth mentioning.

- Under the assumption $\overline{h} = 0$, we have proved that J attains its minium or, equivalently, the problem (Q) admits a weak solution.
- Unlike the problem (P), problem (Q) has no uniqueness of (weak) solution. One can get the existence of another different weak solution by another variaional approach, meaning, a mountain-pass argument (see theorem 18).
- We know from proposition 24 that if $|\overline{h}| > 1$, then there is no solution to (Q).
- To sum up, if $\overline{h} = 0$ there is weak solution to (Q) (not unique) and if $|\overline{h}| > 1$ there is no solution to (Q). What about the case when $|\overline{h}|$ is close to zero, i.e. $|\overline{h}| < \varepsilon$ for some $\varepsilon > 0$ small enough? Well, the answer is not completely clear. Nowadays, this is still an open problem!.

Exercise 29. *In section 3.1, devoted to make a brief introduction to Calculus of variations, we presented the problem*

$$(\star) \quad \begin{cases} x'' = F(x), \ x = (x_1, x_2, x_3) \\ x(0) = P, \ x(1) = Q \end{cases}$$

for some $P, Q \in \mathbb{R}^3$, where $F = -\nabla U$ for some potential $U : \mathbb{R}^3 \to \mathbb{R}$. The exercise consists in showing that if U is upper bounded, then there is (weak) solution to $(\star)$.

► *Hint*: try to minimize the functional $\phi(x) = \int_0^1 \frac{1}{2}\|x'\|^2 - U(x)\ dx$ on the set $\{x \in H^1(0,1)^3 : x(0) = P, x(1) = Q\}$.

3.3 Ekeland variational Principle and min-max theorems

We finish this chapter, and with that this notes, with the statement and proof of the **Ekeland variational Principle**. This principle is known today as a classical theorem that established the basis of the *critical points theory*, more specifically, the *min-max methods* of recent development. Actually, we will underline the connection between the Ekeland variational Principle and the previous section. The principle was discovered by the French mathemtician of Norwegian descent, Ivar Ekeland (1944-), in 1974 when he was associated with the Paris Dauphine University, at the age of 30. The statement and proof that we present here are the same Ekeland did.

Ekeland variational Principle established the basis of the min-max theory. This modern theory is full of results that infer the existence of critical values of a functional from simple geometric conditions of the functional. We will basically just focus on two of this results that have become classical within this theory: *mountain-pass theorem* (due to Ambrosetti and Rabinowitz) and *saddle-point theorem* (due to Rabinowitz). Anyway, if the reader is interested, we suggest to take a look at [3] and [10].

Before presenting these two theorems, it is required some more mathematical knowledge. Particulary, the concept of *Fréchet differentiability*, named after Maurice Fréchet (1878-1973) and *Palais-Smale condition*, named after Richard Palais (1931-) and Stephen Smale (1930-).

Definition 14 (Fréchet differentiability). *Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two normed spaces, $\Omega \subset X$ an open and not-empty subset of X and $x_0 \in \Omega$ a point in Ω, a map $\Phi : \Omega \to Y$ is* ***Fréchet-differentiable*** *at x_0 if, and only if, there exists a continuous linear map $l : X \to Y$ such that*

$$\lim_{h \to 0} \frac{\|\Phi(x_0 + h) - \Phi(x_0) - l(h)\|_Y}{\|h\|_X} = 0$$

We say that Φ is Fréchet differentiable if it is Fréchet differentiable at every point of Ω. The set of functions $\Phi : \Omega \to Y$ which are Fréchet differentiable verifying that the map $x \mapsto \Phi'(x)$, from Ω to $L(X, Y)$, is continuos are denoted by $\mathcal{C}^1(\Omega, Y)$. Note that the only novelty with respect with the differentiability in finite-dimensional spaces is that now we have to demand the continuity to l. A linear map defined on a infinite-dimensional space need not to be continuous! and hence, the Fréchet differentiability in finite dimension coincides with the usual concept of differentiability in $\mathbb{R}^N$.

Recall that $x_* \in X$ is a critical point of $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ when $\Psi'(x_*) = 0$ and $\sigma = \Psi(x_*)$ is a critical level of Ψ. It can be proved the following properties:

- The Fréchet differentiability does not depend on the norms of the two normed spaces, i.e. the concept does not change when norms are replaced by equivalent ones.
- If such l of the definition exists, then is unique and denoted by $l = \Phi'(x_0)$.

- Just like happens with the concept of differentiabilty in $\mathbb{R}^N$, if $\Phi'(x_0)$ exists, then Φ is continuous at x_0.
- Just like happens with the concept of differentiabilty in $\mathbb{R}^N$, if Φ is actually a linear and coninuous map, then is Fréchet differentiable with $\Phi'(x_0) = \Phi$ for every point $x_0 \in \Omega$.

Definition 15 (Palais-Smale sequence)**.** *Given $(X, \|\cdot\|)$ a normed space and a functional $\Psi \in \mathcal{C}^1(X, \mathbb{R})$, a sequence $\{x_n\} \subset X$ is a **Palais-Smale sequence** ($P - S$ in short) if $\{\Psi(x_n)\}$ is bounded and $\Psi'(x_n) \to 0$ in X^{-1} ($\Psi'(x_n)(y) \to 0,\ \forall y \in X$).*

Definition 16 (Palais-Smale sequence at level κ)**.** *Given $(X, \|\cdot\|)$ a normed space and a functional $\Psi \in \mathcal{C}^1(X, \mathbb{R})$, a sequence $\{x_n\} \subset X$ is a **Palais-Smale sequence at level** $\kappa \in \mathbb{R}$ ($P - S_\kappa$ in short) if $\Psi(x_n) \to \kappa$ and $\Psi'(x_n) \to 0$ in X^{-1} ($\Psi'(x_n)(y) \to 0,\ \forall y \in X$).*

Obviously, the fact that $\Psi(x_n) \to \kappa$ does not imply $\Psi'(x_n) \to 0$. Even in dimension $N = 1$, one can construct a $\mathcal{C}^1$-function $F : \mathbb{R} \to \mathbb{R}$ and a sequence $\{x_n\}$ such that $F(x_n) \to \kappa$ for some $\kappa \in \mathbb{R}$ although $F'(x_n) = 1$ for all $n \in \mathbb{N}$.

Definition 17 (Palais-Smale condition)**.** *Given $(X, \|\cdot\|)$ a normed space, a functional $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ has the **Palais-Smale condition** ($(P - S)$ in short) if, and only if, every Palais-Smale sequence $\{x_n\}$ has a converging subsequence.*

Definition 18 (Palais-Smale condition at level κ)**.** *Given $(X, \|\cdot\|)$ a normed space, a functional $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ has the **Palais-Smale condition** at level $\kappa \in \mathbb{R}$ ($(P - S)_\kappa$ in short) if, and only if, every Palais-Smale sequence $\{x_n\}$ at level κ has a converging subsequence.*

It can be easily check the following remarks,

- If $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ has the $(P - S)_\kappa$ condition, then every $P - S_\kappa$ sequence converges (up to a subsequence) to some $x_* \in X$ and, by continuity of Ψ, $\Psi(x_*) = \kappa$ and $\Psi'(x_*) = 0$. In other words, x_* is a critical point of Ψ and κ is a critical level of Ψ. In particular, $\{x \in X : \Psi(x) = \kappa, \Psi'(x) = 0\}$ is relatively compact.
- If $F \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, for some $N \in \mathbb{N}$, is bounded from below and coercive, then F has the $(P - S)$ condition. This is peculiar of the finite dimension: if $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ and X is infinite-dimensional, Ψ might be bounded from below and coercive without having the $(P - S)$ condition. Can you think in an example of this?

Exercise 30. *Find a functional defined on an infinite-dimensional Banach space that is bounded from below and coercive, but $(P - S)_\kappa$ does not hold for some $\kappa \in \mathbb{R}$.*

Theorem 17 (Ekeland variational Principle)**.** *Let (X, d) be a complete metric space, $u \in X$ an arbitrary fixed point of X, $\varepsilon > 0$ an arbitrary fixed positive number and $\psi : X \to \mathbb{R} \cup \{+\infty\}$ a functional on X that is bounded below and not identically equal to $+\infty$, then for each $\lambda > 0$, there exists $v \in X$ such that*

(i) $\psi(v) \leq \psi(u)$

(ii) $d(u,v) \leq \dfrac{1}{\lambda}$

(iii) $\psi(v) < \psi(w) + \varepsilon\lambda d(v,w) \quad \forall w \in X, w \neq u$

Remark 18. *The Ekeland variational Principle asserts that there exists nearly optimal solutions to some optimization problems, meaning, v is not a minimizer of ψ on X, instead is the minimizer of a perturbed problem concerning ψ.*

Proof. Without loss of generality, one can restrinct to the case $\lambda = 1$, otherwise consider the new distance $\tilde{d}(x,y) = \lambda d(x,y)$ for all $x, y \in X$. Firstly, we define a partial order in X as follows:

$$x \leq y \overset{def}{\Longleftrightarrow} \psi(x) + \varepsilon d(x,y) \leq \psi(y) \qquad \forall x, y \in X$$

It is straightforward that this relation in X is an order relation, that is, is reflexive ($x \leq x$), antisymmetric ($x \leq y,\ y \leq x \Rightarrow x = y$) and transitive ($x \leq y,\ y \leq z \Rightarrow x \leq z$). It can be easily checked by using the properties of the distance map d. Secondly, we define by induction a sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ as follows: take $u_1 = u$ and set $S_1 = \{w \in X : w \leq u_1$ then, given $u_n \in S_n$, take $u_{n+1} \in S_n$ with $\psi(u_{n+1}) \leq \inf_{S_n} \psi + \frac{1}{n}$ (this is always possible by the characterization of infimum points).

This sequence os sets $\{S_n\}$ enjoys the following properties for all integer n,

- $S_n \neq \emptyset$ because $u_n \in S_n$.
- $S_{n+1} \subset S_n$ owing to the definition itself.
- S_n is closed due to the lower semicontinuity of ψ and the continuity of d: if $\{x_n\} \subset S_n$ and $x_k \to x$, then $x_k \leq u_n$ for all $k \in \mathbb{N}$ and thus

  $$\psi(x_k) \leq \psi(u_n) - \varepsilon d(x_k, u_n)$$

  taking lower limit,

  $$\liminf_{k\to\infty} \psi(x_k) \leq \liminf_{k\to\infty} \psi(u_n) - \varepsilon d(x_k, u_n)$$

  using the lower semicontinuity of ψ and the continuity of d,

  $$\psi(x) \leq \psi(u_n) - \varepsilon d(x, u_n)$$

  which is nothing but $x \in S_n$.
- $diam(S_n) \to 0$ as $n \to \infty$: take $w \in S_{n+1}$, then $\psi(w) + \varepsilon d(w, u_{n+1}) \leq \psi(u_{n+1})$, but also $w \in S_n$ so $\psi(w) \geq \inf_{S_n} \psi$ and

  $$\begin{aligned} d(w, u_{n+1}) &\leq \frac{1}{\varepsilon}[\psi(u_{n+1} - \psi(w)] \leq \frac{1}{\varepsilon}\left[\inf_{S_n} \psi + \frac{1}{n} - \psi(w)\right] \leq \\ &\leq \frac{1}{\varepsilon}\left[\inf_{S_n} \psi + \frac{1}{n} - \inf_{S_n} \psi\right] = \frac{1}{\varepsilon n} \end{aligned}$$

  From this computation follows $diam(S_n) \to 0$ as $n \to \infty$.

Finally, the completeness of X and this four previous assertions ensure that the intersection of all the sets $\{S_n\}$ is formed by only one point (this is often referred as Cantor's intersection theorem), say $v \in X$,

$$\bigcap_{n\in\mathbb{N}} S_n = \{v\}$$

This point v satisfies the three conditions $(i) - (ii) - (iii)$ of the theorem. Indeed,

(i): $\psi(v) \leq \psi(u)$

$$\bigcap_{n\in\mathbb{N}} S_n = \{v\} \Rightarrow v \in S_1 \Rightarrow v \leq u_1 := u \Rightarrow \psi(v) + \underbrace{\varepsilon d(v,u)}_{\geq 0} \leq \psi(v) \Rightarrow \psi(v) \leq \psi(u)$$

(ii): $d(u,v) \leq 1$ (remember that $\lambda = 1$)

$$\psi(v) + \varepsilon d(u,v) \leq \psi(u) \Leftrightarrow \varepsilon d(u,v) \leq \psi(u) - \psi(v);$$

$$\varepsilon d(u,v) \leq \inf \psi + \varepsilon - \psi(v) \leq \inf \psi + \varepsilon - \inf \psi = \varepsilon \Rightarrow d(u,v) \leq 1$$

(iii): $\psi(v) < \psi(w) + \varepsilon d(v,w),\ w \in X, w \neq u$ (again $\lambda = 1$)
It is imposible that $w \leq v$, otherwise w would belong to the intersection of all the sets $\{S_n\}$, so $w \nleq v$ which means $\psi(v) < \psi(w) + \varepsilon d(v,w)$. □

Remark 19. *Ekeland's principle has been shown by F. Sullivan to be equivalent to the completeness of metric spaces.*

The next result is a consequence of Ekeland variational Principle and, under a geometric assumption, guarantees the existence of a minimizing sequence that is close to another minimizing one (fixed at the beginning) and with a new property more, which can give us useful information.

Corollary 7. *If (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}$ a $\mathcal{C}^1$-function that is bounded from below and $\{x_n\} \subset X$ is a minimizing sequence for φ on X, then there exists a sequence $\{y_n\} \subset X$ with*

1. $\varphi(y_n) \to \inf_X \varphi$ *(as $n \to \infty$)*
2. $\|y_n - x_n\| \to 0$ *(as $n \to \infty$)*
3. $\|\varphi'(y_n)\| \to 0$ *(as $n \to \infty$)*

Proof. If $\varphi(x_n) \equiv \inf_X \varphi$ the proof is done. Otherwise, set $\varepsilon_n = \varphi(x_n) - \inf_X \varphi > 0$ and $\lambda_n = 1/\sqrt{\varepsilon_n}$ for all $n \in \mathbb{N}$. After applying Ekeland variational Principle, and yields $y_n \in X$ for each λ_n such that

1.
$$\inf_X \varphi \leq \varphi(y_n) \leq \varphi(x_n) \to \inf_X \varphi \Rightarrow \varphi(y_n) \to \inf_X \varphi$$

2.
$$\|y_n - x_n\| = d(x_n, y_n) \leq \frac{1}{\sqrt{\lambda_n}} = \sqrt{\varepsilon_n} \to 0$$

3. Take $h \in X$ with $\|h\|_X = 1$. Since $\varphi \in \mathcal{C}^1(X)$, it must be

$$\begin{aligned}
\varphi'(y_n)(h) &= \lim_{t\to 0^+} \frac{\varphi(y_n + th) - \varphi(y_n)}{t} \geq \\
&\geq \lim_{t\to 0^+} \frac{\varphi(y_n) - \sqrt{\varepsilon_n}t - \varphi(y_n)}{t} = -\sqrt{\varepsilon_n} \\
\varphi'(y_n)(h) &= \lim_{t\to 0^-} \frac{\varphi(y_n + th) - \varphi(y_n)}{t} \leq \\
&\leq \lim_{t\to 0^-} \frac{\varphi(y_n) + \sqrt{\varepsilon_n}t - \varphi(y_n)}{t} = +\sqrt{\varepsilon_n}
\end{aligned}$$

and hence
$$0 \leq \|\varphi'(y_n)(h)| \leq \sqrt{\varepsilon_n} \quad \forall h \in X, \|h\|_X = 1;$$
$$\|\varphi'(y_n)\| = \sup_{\|h\|_X=1} |\varphi'(y_n)(h)| \leq \sup_{\|h\|_X=1} \sqrt{\varepsilon_n} = \sqrt{\varepsilon_n} \to 0$$

□

Exercise 31. *Prove that if (X, d) is a complete metric space and $\varphi : X \to \mathbb{R}$ a $\mathcal{C}^2$-function that is bounded from below and $\|D^2\varphi\| \leq C$ for some $C > 0$, then every minimizing sequence of φ on X is a Palais-Smale sequence (at level $\inf_X \varphi$).*

Ekeland variational Principle was one of the first theorem of a large list of results that have the same purpose: establish the existence of critical values or level of functionals knowing in advance some geometrical conditions. This field in Nonlinear Analysis is called nowadays *min-max theory.* As it was said at the beginning of this section, here it is only presented two more result, one by Ambrosetti and Rabinowitz and another one by Rabinowitz.

Theorem 18 (Mountain-pass, Ambrosetti & Rabinowitz, 1973)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ having the Palais-Smale condition at any level $\kappa \in \mathbb{R}$ and*

(MP1) $\Phi(0) = 0$

(MP2) $\exists\ \varepsilon, \delta > 0 : \|x\| = \varepsilon \Rightarrow \Phi(x) \geq \delta$

(MP3) $\exists\ y \in X,\ \|y\| > \varepsilon : \Phi(y) \leq 0$

then
$$\mathfrak{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \delta$$
is a critical level of Φ, meaning there exists $z \in X$ with $\Phi'(z) = 0$ and $\Phi(z) = \mathfrak{c}$, where Γ is the family of continuous curves on X joining 0 and y, that is
$$\Gamma = \{\gamma : [0, 1] \to X : \gamma \text{ continuous }, \gamma(0) = 0, \gamma(1) = y\}$$

The name of mountain-pass theorem is justified by its geometric interpretation: imagine that the Earth is modeled by $\mathbb{R}^2$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}$ expresses the height of the point $x \in \mathbb{R}^2$ over the sea level, then $(MP1) - (MP2) - (MP3)$ are fulfilled if, and only if, the origin is located in a valley surrounded by mountains and, far away from the mountains, there is a point with negative height, that is, the sea!.

Theorem 19 (Saddle-point, Rabinowitz, 1984). *Let $(X, \|\cdot\|)$ be a Banach space and $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ having the Palais-Smale condition at any level $\kappa \in \mathbb{R}$ and*

(SP1) $X = V \oplus V^\perp$ *with* $0 < \dim V < \infty$

(SP2) $\rho := \inf_X \Phi(x) \in \mathbb{R}$

(SP3) there exists $U \subset V$ an open and bounded neighbour of 0 in V with $\sup_{\partial U} \Phi(x) < \rho$

then

$$\mathfrak{c} = \inf_{\lambda \in \Lambda} \max_{x \in \overline{U}} \Phi(\lambda(x)) \geq \rho$$

is a critical level of Φ, meaning there exists $z \in X$ with $\Phi'(z) = 0$ and $\Phi(z) = \mathfrak{c}$, where Λ is the family of continuous functions from $\overline{U}$ to X that fix the boundary ∂U, that is

$$\Lambda = \{\lambda : \overline{U} \to X : \lambda \text{ continuous }, \lambda(x) = x, \ \forall x \in \partial U\}$$

Once again the name of saddle-point comes from a geometric argument. In general, the type of funtionals which Saddle-point theorem can be applied to, respond to the following geometry: Φ is concave in V, convex in X and obey a coercivity condition at infinity.

Although these theorems can be derived from the Ekeland variational Principle, it is preferred to prove them using the *deformation lemma* whose proof will be omitted here. Before stating the deformation lemma, let us make two definitions or, better said, notation.

Given $(X, \|\cdot\|)$ be a Banach space, $\Phi : X \to \mathbb{R}$ a functional and $c \in \mathbb{R}$, denote

$$Z_c = \{x \in X : \Phi(x) \leq c\}$$

$$K_c = \{x \in X : \Phi(x) = c, \Phi'(x) = 0\}$$

When $K_c \neq \emptyset$, we say that c is a critical level of Φ.

Lemma 5 (Deformation). *Let $(X, \|\cdot\|)$ be a Banach space, $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ having the Palais-Smale condition at some level $c \in \mathbb{R}$, then for all $\varepsilon_0 > 0$, there is $\delta_0 \in (0, \varepsilon_0)$ and $\eta \in C([0,1] \times X, X)$ such that*

(a) $\eta(0, x) = x$ *for all* $x \in X$

(b) $\eta(1, x) = x$ *for all* $x \in X$ *with* $\Phi(x) \notin [c - \varepsilon_0, c + \varepsilon_0]$

(c) $\eta(1, Z_{c+\delta_0}) \subset Z_{c-\delta_0}$

Remark 20. *For each fixed $t \in [0,1]$, the map $\eta(t,\cdot) : X \to X$ is an homeomorphism from X onto itself. Moving the variable t in $[0,1]$, this can be seen as a family of continuous maps that are the identity on X when $t = 0$ and continuously deform the space X while t goes through the interval $[0,1]$. Part (a) claims that $\eta(0,\cdot)$ is the identity map on X. Part (b) asserts that also $\eta(1,\cdot)$ is the identity map but only when x is not near of c in the sense that $|x-c| \geq \delta$. Part (c) assures that $\eta(1,\cdot)$ transforms $Z_{c+\varepsilon}$ into another subset (of lower level) contents in $Z_{c-\varepsilon}$. This last property turns fundamental in the proofs of several min-max theorems.*

Let us write the proofs of both theorems, which can be found in [10].

Proof of mountain-pass theorem. If $\Upsilon = \{\gamma[0,1] : \gamma \in \Gamma\}$, then $\mathfrak{c} = \inf_{W\in\Upsilon} \max_{x\in W} \Phi(x)$. Clearly, $\Gamma \neq \emptyset$ ($\gamma : [0,1] \to X,\ t \mapsto \gamma(t) = ty$ belongs to Γ) which implies that $\Upsilon \neq \emptyset$. Since elements of Υ are compact subsets of X, for each $W \in \Upsilon$ there exists $\sup_{x\in W} \Phi(x)$. In addition, for each $W \in \Upsilon$, there is (at least) one element $x \in W$ with $\|x\| = \varepsilon$. Indeed, this follows from the continuity of γ and $\|\gamma(0)\| = 0$ and $\|\gamma(1)\| > \varepsilon$ by $(MP3)$ and so $\mathfrak{c} \geq \delta$ by $(MP2)$. Finally, if $K_c = \emptyset$, take $\varepsilon_0 = \delta/2$ in the deformation lemma and set $\delta_0 > 0$ and $\eta \in \mathcal{C}([0,1] \times X, X)$ provided by the lemma. By the fact that $\mathfrak{c} = \inf_{W\in\Upsilon} \max_{x\in W} \Phi(x)$, there must be $W \in \Upsilon$ such that

$$\sup_{x\in W} \Phi(x) \leq c + \delta_0 \Rightarrow W \subset Z_{c+\delta_0} \Rightarrow \eta(1,W) \subset Z_{c-\delta_0}$$

However, it can be checked that $\eta(1,W) \in \Upsilon$, and

$$\sup_{x\in\eta(1,W)} \Phi(x) \leq \sup_{x\in Z_{c-\delta_0}} \Phi(x) \leq c - \delta_0$$

and this contradicts the definition $\mathfrak{c} = \inf_{W\in\Upsilon} \sup_{x\in W} \Phi(x)$, therefore there exists $z_* \in X$ with $\Phi(z_*) = \mathfrak{c} \geq \delta$ and $\Phi'(z_*) = 0$. This completes the proof. □

Proof of saddle-point theorem. If $\Upsilon = \{\lambda(\overline{U}) : \lambda \in \Lambda\}$, then $\mathfrak{c} = \inf_{W\in\Upsilon} \max_{x\in W} \Phi(x)$. Clearly, $\Lambda \neq \emptyset$ (the identity restrincted to $\overline{U}$ belongs to Υ) which implies that $\Upsilon \neq \emptyset$. Since elements of Υ are compact subsets of X, for each $K \in \Upsilon$ there exists $\sup_{x\in K} \Phi(x)$. In addition, for each $W \in \Upsilon$, there is (at least) one element $x \in W$ with $\lambda(x) \in X$. Indeed, if $\lambda \in \Lambda$ and $\pi : X \to V$ is the correponding projection map of X onto V, the map $\pi\circ\lambda : \overline{U} \to V$ is continuous by the chain rule and it does not vanish on the boundary of $\overline{U}$ owing to $(\pi \circ \lambda)(x) = x$ for every $x \in \partial U$ and thus, by the Brouwer degree, $\deg(\pi \circ \lambda, U, 0) = 1 \neq 0$, so there exists $x \in U$ with $(\pi \circ \lambda)(x) = 0$, i.e. $\lambda(x) \in X$, and hence $\mathfrak{c} \geq \rho$. Finally, if $K_c = \emptyset$, take $\varepsilon_0 = \rho/2$ in the deformation lemma and set $\delta_0 > 0$ and $\eta \in \mathcal{C}([0,1] \times X, X)$ provided by the lemma. By the fact that $\mathfrak{c} = \inf_{W\in\Upsilon} \sup_{x\in W} \Phi(x)$, there must be $W \in \Upsilon$ such that

$$\sup_{x\in W} \Phi(x) \leq c + \delta_0 \Rightarrow W \subset Z_{c+\delta_0} \Rightarrow \eta(1,W) \subset Z_{c-\delta_0}$$

However, it can be checked that $\eta(1,W) \in \Upsilon$, and

$$\sup_{x\in\eta(1,W)} \Phi(x) \leq \sup_{x\in Z_{c-\delta_0}} \Phi(x) \leq c - \delta_0$$

and this contradicts the definition $\mathfrak{c} = \inf_{W\in\Upsilon} \sup_{x\in W} \Phi(x)$, therefore there exists $z_* \in X$ with $\Phi(z_*) = \mathfrak{c} \geq \rho$ and $\Phi'(z_*) = 0$. This completes the proof. □

Exercise 32. *Study the existence of weak solution of the toy-problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ and $1 < p < \frac{N+2}{N-2}$ if $N > 2$ or $p = +\infty$ if $N = 2$, using the mountain-pass theorem. Is the obtained solution trivial?

If one takes a look carefully at both proofs, it can be easily observed the enormous analogy among them. This turns out to be a genereal philosophy of the min-max theorems. Actually, more abstract theorems on this line are already proved (see [3] or [11]). It is convenient to underline that the method of the proofs is the same in the majority of cases. The essential idea in both proofes is definition of the family Υ of subsets of X. This family Υ must be (positively) *η-invariant*, namely $\eta(t, W) \in \Upsilon$ for all $W \in \Upsilon$ and for all $t \in [0, 1]$.

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