Existence, symmetry and stability results for some quasi-linear elliptic equations

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Talk based upon the recent papers:

L. JEANJEAN, M.S.,
Existence and symmetry of least energy solutions for a class of quasi-linear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1701–1716.

M. COLIN, L. JEANJEAN, M.S.,
Stability and instability results for standing waves of quasi-linear Schrodinger equations, *preprint*, 40pp, arXiv: 0906.5261.

L. JEANJEAN, M.S.,
An approach to minimization under constraint: the added mass technique, *preprint*, 23pp, arXiv: 0906.1081

H. HAJAIEJ, M.S.,
Generalized Polya-Szegö inequality and applications to some quasi-linear problems, *preprint*, 21pp, arXiv:0903.3975v3.

Some classical related references:

H. Berestycki, P.-L. Lions,
Nonlinear scalar field equations. I. Existence of a ground state,
Arch. Rational Mech. Anal. **82** (1983), 313–345.

H. Brezis, E.H. Lieb,
Minimum action solutions of some vector field equations,
Comm. Math. Phys. **96** (1984), 97–113.

J.E. Brothers, W.P. Ziemer,
Minimal rearrangements of Sobolev functions,
J. Reine Angew. Math. **384** (1988), 153–179.

T. Cazenave, P.L. Lions,
Orbital stability of standing waves for some nonlinear Schrödinger equations,
Comm. Math. Phys. **85** (1982), 549–561.

Main goals I

Show

existence and **radial symmetry**

of **any** *least energy solution* to

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

We look for solutions in $D^{1,p}(\mathbb{R}^n)$, $1 < p \leq n$. If $F(s) = \int_0^s f(t)$, the equation is formally associated with the functional

$$I(u) = \int_{\mathbb{R}^n} j(u, Du) - \int_{\mathbb{R}^n} F(u).$$

A least energy solution is nontrivial function $u \in D^{1,p}(\mathbb{R}^n)$ with

$$I(u) = \inf \left\{ I(v) : v \in D^{1,p}(\mathbb{R}^n) \setminus \{0\} \text{ is a solution of the eq.} \right\}.$$

Main goals IIa

Improve and provide new simplified proofs for

generalized Polya-Szegö inequalities

$$\int_{\mathbb{R}^n} j(u^*, |\nabla u^*|)dx \leq \int_{\mathbb{R}^n} j(u, |\nabla u|)dx,$$

$$\int_{\mathbb{R}^n} F(|x|, u_1, \ldots, u_m)dx \leq \int_{\mathbb{R}^n} F(|x|, u_1^*, \ldots, u_m^*)dx$$

Here u^* denotes the Schwarz symmetrization of u.

- explicit dependence on u (first inequality)
- multiple components and x dependence (second inequality)

Assumptions? Impact on applications?

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Main goals IIb

Better understanding and new simplified proofs for

identity cases, that is

$$\int_{\mathbb{R}^n} j(u^*, |\nabla u^*|)dx = \int_{\mathbb{R}^n} j(u, |\nabla u|)dx, \quad \mathcal{L}^n(\{x : \nabla u^*(x)\}) = 0$$

under strict convexity of $\{t \mapsto j(s,t)\}$ imply that

$$u(x) = u^*(x - x_0), \quad x \in \mathbb{R}^n.$$

Applications: to show that **any** minimizer of a variational problem is radially symmetric.

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Applications: to show that **any** minimizer of a variational problem is radially symmetric.

Main goals III

Radial symmetry of minimax CP (J. Van Schaftingen in 2005 for C^1 case)

Theorem

Let X be a Banach spaces, $S \subset X$, $*$ the Schwarz symmetrization. Let $f : X \to \mathbb{R}$ a continuous functional, M be a metric space and M_0 a closed subset of M and $\Gamma_0 \subset C(M_0, X)$. Let $\Gamma = \{\gamma \in C(M, X) : \ \gamma|_{M_0} \in \Gamma_0\}$,

$$\infty < c = \inf_{\gamma\in\Gamma} \sup_{\tau\in M} f(\gamma(\tau)) > \sup_{\gamma_0\in\Gamma_0} \sup_{\tau\in M_0} f(\gamma_0(\tau)) = a,$$

and that for all polarized H and $u \in S$, we have $f(u^H) \le f(u)$. Then, for every $\varepsilon \in (0,(c-a)/2)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau\in M} f(\gamma(\tau)) < c+\varepsilon, \quad \gamma(M) \subset S, \quad \gamma|_{M_0}^{H_0} \in \Gamma_0 \text{ for some } H_0 \in \mathcal{H}_*,$$

there exists $u \in X$ such that

$$c-2\varepsilon \le f(u) \le c+2\varepsilon, \quad |df|(u) \le 8\varepsilon/\delta, \quad \|u-u^*\|_V \le K\delta.$$

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Main goals III

Roughly speaking, if the functional does not increase under polarization, then the deformation Lemma provides almost critical points which are almost Schwarz symmetric. In the limit one finds a Schwarz symmetric critical point. For instance, one can apply these kind of result to a functional like

$$f(u) = \int_{B_1} j(u, |\nabla u|)dx - \int_{B_1} G(|x|, u)dx,$$

where the growth condition on j allow j to be unbounded with respect to u. In these cases the functional is merely lower semicontinuous, and nonsmooth critical point theory has been applied in my paper

B. Pellacci, M.S.,
Unbounded critical points for a class of lower semicontinuous functionals,
J. Differential Equations **201** (2004), 25–62.

With the new symmetric statement, under suitable assumption I now find a radially symmetric mountain pass solution as a critical point of f (in the sense of weak slope).

Main goals IV

Quasi-linear Schrödinger equation

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

For this equation, investigate

- property of ground states;
- stability;
- instability,
- bifurcations results.

The principal part of the Lagrangian associated with the stationary problem is $j(s,\xi) = \frac{1}{2}(1+s^2)|\xi|^2$.

Know existence and symmetry results

Existence and symmetry of least energy solution (scalar case) for

$$-\Delta_p u = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

$p = 2$: Berestycki, Lions ('83).
$p \neq 2$: Gazzola, Ferrero, Tang, Serrin, Ni, Peletier, Atkinson, Franchi, Lanconelli, Citti, and others..

Case $p = 2$ and some studies of the case $p \neq 2$ use constrained minimization (suitable assumptions on f, F):

$J|_M(u) = \int_{\mathbb{R}^n} |\nabla u|^p, \quad M = \{u \in D^{1,p} : \int_{\mathbb{R}^n} F(u) = 1\}$. Of course, rearrangement inequalities can be used here!

$\int_{\mathbb{R}^n} |\nabla u^*|^p \leq \int_{\mathbb{R}^n} |\nabla u|^p, \quad \int_{\mathbb{R}^n} F(u^*) = \int_{\mathbb{R}^n} F(u).$

Existence: OK. Radial symmetry: OK.

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Existence of least energy solution (vector case) for

$$-\Delta u_i = f(u_1, \dots, u_m) \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

H. Brezis, E.H. Lieb (1984).

This paper develops a new technique, a refinement of the constrained minimization technique. In fact, in general, unless one assumes some cooperativity conditions (see later) on F

$$\int_{\mathbb{R}^n} F(u_1, \dots, u_m)dx \not\leq \int_{\mathbb{R}^n} F(u_1^*, \dots, u_m^*)dx$$

Hence, no rearrangement technique!

Existence: OK. Radial symmetry: left open.

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Recall of cooperativity conditions

For $F : [0, \infty) \times \mathbb{R}^m_+ \to \mathbb{R}$ measurable in r and continuous with respect to $(s_1, \dots, s_m)$ with $F(r, 0, \dots, 0) = 0$ for any r,

$$F(r, s + he_i + ke_j) + F(r, s) \geq F(r, s + he_i) + F(r, s + ke_j),$$
$$F(r_1, s + he_i) + F(r_0, s) \leq F(r_1, s) + F(r_0, s + he_i),$$

for every $i \neq j = 1, \dots, m$ where e_i denotes the i-th standard basis vector in $\mathbb{R}^m$, $r > 0$, $h, k > 0$, $s \in \mathbb{R}^m$ and r_0, r_1 with $0 < r_0 < r_1$.

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General Conditions

Introducing the functionals,

$$J(u) = \int_{\mathbb{R}^n} j(u, Du), \quad V(u) = \int_{\mathbb{R}^n} F(u), \qquad u \in D^{1,p}(\mathbb{R}^n),$$

we consider the following constrained problem

$$\text{minimize } J(u) \text{ subject to the constraint } V(u) = 1. \qquad (P_1)$$

More precisely, let us set

$$X = \{u \in D^{1,p}(\mathbb{R}^n) : F(u) \in L^1(\mathbb{R}^n)\},$$

and

$$T = \inf_{\mathcal{C}} J, \qquad \mathcal{C} = \{u \in X : V(u) = 1\}.$$

General Conditions

Consider *3 conditions* (**Existence, Euler Eq., Pohozaev Id.**):

(C1) $T > 0$ and problem (P_1) has a minimizer $u \in X$;

(C2) any minimizer $u \in X$ of (P_1) is C^1 solution and it satisfies

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = \mu f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

for some $\mu \in \mathbb{R}$.

(C3) any solution $u \in X$ of the previous equation satisfies

$$(n - p)J(u) = \mu n V(u).$$

Of course, usually, condition **(C1)** is the *more delicate* to check.

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General Conditions

Theorem

Assume that $1 < p < n$ *and that conditions* **(C1)-(C3)** *hold. Then, under suitable assumptions on* f, F *(see in a minute..), the problem*

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

admits a least energy solution and each least energy solution has a constant sign and is radially symmetric, up to a translation in $\mathbb{R}^n$*.*

J. Byeon, L. Jeanjean, M. Maris,
Symmetry and monotonicity of least energy solutions,
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Main goals

- Our aim is to look for conditions on j and f, F such that conditions **(C1)**, **(C2)** and **(C3)** are fulfilled.

- We do not involve Schwarz symmetrization arguments, although there are some results for the operator $j(u, |\nabla u|)$. Some results can be obtained for systems (this avoids cooperativity conditions on F);

- Existence proofs follows the line of BREZIS-LIEB, CMP **96** ('84);

- Nearly optimal assumptions on F (some improvements also with respect to the literature of the p-laplacian case, $j(\xi) = |\xi|^p$).

- Radial symmetry relies on the paper by MARIŞ, On the symmetry of minimizers, *Arch. Rat. Mech. Anal.*, (2009).

Main result: assumptions on F

Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $F(0) = 0$.

$$\limsup_{s\to 0} \frac{F(s)}{|s|^{p^*}} \leq 0;$$

$$\text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0.$$

Moreover, if $f(s) = F'(s)$ for any $s \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(s)}{|s|^{p^*-1}} = 0.$$

Main result: assumptions on j

Let $j(s,\xi) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 in s and ξ and denote by j_s and j_ξ the derivatives of j with respect of s and ξ.

$$\{\xi \mapsto j(s,\xi)\} \text{ is strictly convex and } p\text{-homogeneous.}$$

There exist positive constants c_1, c_2, c_3, c_4 and R with

$$c_1|\xi|^p \leq j(s,\xi) \leq c_2|\xi|^p;$$

$$|j_s(s,\xi)| \leq c_3|\xi|^p, \quad |j_\xi(s,\xi)| \leq c_4|\xi|^{p-1};$$

$$j_s(s,\xi)s \geq 0, \qquad \text{for } |s| \geq R.$$

Main result: statement for $p < n$

Theorem

Equation

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

admits a radially symmetric least energy solution $u \in D^{1,p}(\mathbb{R}^n)$.

Theorem

Any least energy solution of

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

has a constant sign and is radially symmetric.

Main result: statement for $p = n$

Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $F(0) = 0$. We assume:

$$\text{there exists } \delta > 0 \text{ such that } F(s) < 0 \text{ for all } 0 < |s| \leq \delta;$$

$$\text{there exists } s_0 \in \mathbb{R} \text{ such that } F(s_0) > 0;$$

$$\text{there exist } q > 1 \text{ and } c > 0 \text{ such that } |f(s)| \leq c + c|s|^{q-1} \text{ for all } s \in \mathbb{R}.$$

$$\text{if } u \in D^{1,n}(\mathbb{R}^n) \text{ and } u \not\equiv 0 \text{ then } f(u) \not\equiv 0.$$

Theorem

Equation

$$-\mathrm{div}(j_\xi(u, Du)) + j_s(u, Du) = f(u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

admits a least energy solution $u \in D^{1,n}(\mathbb{R}^n)$. Furthermore any least energy solution has a constant sign and if a least energy solution $u \in D^{1,n}(\mathbb{R}^n)$ satisfies $u(x) \to 0$ as $|x| \to \infty$ it is radially symmetric, up to a translation in $\mathbb{R}^n$.

Sketch of the Proof (just for case $p < n$)

Consider a minimizing sequence $(u_h) \subset \mathcal{C}$ for J, $F(u_h) \in L^1(\mathbb{R}^n)$,

$$J(u_h) = \lim_h \int_{\mathbb{R}^n} j(u_h, Du_h) = T, \quad V(u_h) = \int_{\mathbb{R}^n} F(u_h) = 1.$$

Hence (u_h) is bounded, goes weakly to u in $D^{1,p}$ and, by convexity,

$$\int_{\mathbb{R}^n} j(u, Du) \leq \liminf_h \int_{\mathbb{R}^n} j(u_h, Du_h) = T.$$

Working on the assumptions on F, we find $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\mathcal{L}^n\big(\{x \in \mathbb{R}^n : |u_h(x)| > \varepsilon_1\}\big) \geq \varepsilon_2, \qquad \text{for all } h \in \mathbb{N}.$$

Hence, by a lemma due to Lieb, there exists a shifting sequence $(\xi_h) \subset \mathbb{R}^n$ such that $(u_h(x + \xi_h))$ converges weakly to a nontrivial limit. Thus, we may assume that $u \not\equiv 0$.

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This follows from a useful property, Lemma 6, from

E.H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, *Invent. Math.* **74** (1983), 441–448.

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Semi-linear case does not require local strong convergence in $D^{1,p}$.

(Fully) quasi-linear case requires local strong convergence in $D^{1,p}$.

1. We use Ekeland variational principle (in nonsmooth framework);

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Sketch of the Proof

6. Take a $\phi \in D^{1,p}$ with *compact support* and

$$1 + \int_{\mathbb{R}^n} F(u+\phi) - F(u) > 0.$$

Then,

$$\int_{\mathbb{R}^n} F(u_h+\phi) = 1 + \int_{\mathbb{R}^n} F(u+\phi) - \int_{\mathbb{R}^n} F(u) + o(1).$$

as $h \to \infty$.

7. Moreover (here we need the local strong convergence),

$$\int_{\mathbb{R}^n} j(u_h+\phi, Du_h + D\phi) = T + \int_{\mathbb{R}^n} j(u+\phi, Du+D\phi) - j(u, Du) + o(1),$$

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6. Take a $\phi \in D^{1,p}$ with *compact support* and

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as $h \to \infty$.

Sketch of the Proof

8. Hence, choosing $w = u_h + \phi$ above, and taking $h \to \infty$,

$$T + \int_{\mathbb{R}^n} j(u + \phi, Du + D\phi) - \int_{\mathbb{R}^n} j(u, Du) \geq T \left(1 + \int_{\mathbb{R}^n} F(u + \phi) - \int_{\mathbb{R}^n} F(u)\right)^{\frac{n-p}{n}},$$

for any such a $\phi \in D^{1,p}$ with compact support.

9. Fixed λ close to 1, for some $r > 1$ consider $\Lambda \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$,

$$\Lambda(t) = \lambda \quad \text{if } t \leq 1, \quad \Lambda(t) = 1 \text{ if } t \geq r,$$

with $\rho = \inf_{\mathbb{R}^+} \Lambda > \frac{1}{2}$ and $\sup_{\mathbb{R}^+} |\Lambda'| < \frac{\rho}{r}$.

10. We consider $\Pi_h : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Pi_h(x) = h\Pi\left(\frac{x}{h}\right) = \begin{cases} \lambda x & \text{if } |x| \leq h, \\ \Lambda\left(\frac{|x|}{h}\right)x & \text{if } h \leq |x| \leq rh, \\ x & \text{if } |x| \geq rh, \end{cases}$$

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$$\phi_h(x) = u(\Pi_h(x)) - u(x).$$

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Sketch of the Proof

11. Hence $\phi_h \in D^{1,p}(\mathbb{R}^n)$ has compact support and

$$1 + \int_{\mathbb{R}^n} F(u + \phi_h) - F(u) > 0,$$

at least for all values of λ sufficiently close to 1.

Hence, for any $h \in \mathbb{N}$, we conclude

$$T + \int_{\mathbb{R}^n} j(u + \phi_h, Du + D\phi_h) - \int_{\mathbb{R}^n} j(u, Du)$$
$$\geq T\left(1 + \int_{\mathbb{R}^n} F(u + \phi_h) - \int_{\mathbb{R}^n} F(u)\right)^{\frac{n-p}{n}}.$$

Sketch of the Proof

12. It also holds

$$\int_{\mathbb{R}^n} j(u+\phi_h, Du + D\phi_h) = \lambda^{p-n} \int_{\mathbb{R}^n} j(u, Du) + o(1),$$
$$\int_{\mathbb{R}^n} F(u+\phi_h) = \lambda^{-n} \int_{\mathbb{R}^n} F(u) + o(1),$$

as $h \to \infty$. Then

$$T + (\lambda^{p-n} - 1) \int_{\mathbb{R}^n} j(u, Du) \geq T \left(1 + (\lambda^{-n} - 1) \int_{\mathbb{R}^n} F(u)\right)^{\frac{n-p}{n}}$$

for every λ sufficiently close to 1. Choosing $\lambda = 1 \pm \omega$ with $\omega > 0$ small and then letting $\omega \to 0^+$, we conclude that

$$\int_{\mathbb{R}^n} j(u, Du) = T \int_{\mathbb{R}^n} F(u).$$

Sketch of the Proof

This forces immediately

$$T = \int_{\mathbb{R}^n} j(u, Du), \qquad \int_{\mathbb{R}^n} F(u) = 1,$$

concluding the proof!

Other directions of work:

Generalized Polya-Szegö inequality:

Theorem

Whenever $\{\xi \mapsto j(s, |\xi|)\}$ is convex and the associated functional of the calculus of variation is weakly lower semicontinuous,

$$\int_{\mathbb{R}^n} j(u^*, |\nabla u^*|)dx \leq \int_{\mathbb{R}^n} j(u, |\nabla u|)dx,$$

allowing minimizers of some variational pb to be radial. No restrictive growth conditions is needed (some previous results by Tahraoui). Preprint with H. Hajaiej includes various applications.

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Other directions of work:

Idea: denoting

$$J(u) = \int_{\mathbb{R}^n} j(u, |\nabla u|)dx,$$

given $u \in W^{1,p}_+(\mathbb{R}^N)$, prove that there exists a sequence (u_n) with

$$J(u_{n+1}) \leq J(u_n) \leq \cdots \leq J(u), \qquad u_n \rightharpoonup u^*.$$

Then weakly lower semicontinuity yields the assertion. End of the proof!

For a dense sequence $(H_n)_{n\geq 1}$ in a half plane H, define

$$u_{n+1} = ((u_0^{H_1})^{H_2})^{\dots H_{n+1}}, \quad u_0 = u.$$

where

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^N \setminus H. \end{cases}$$

Two-point polarization of u ($\sigma_H(x)$ a reflection of x w.r.t. H).

Other directions of work:

Identity cases in the generalized Polya-Szegö inequality:

Theorem

$$M = \mathrm{esssup}_{\mathbb{R}^N} u, \quad C^* = \{x \in \mathbb{R}^N : \nabla u^*(x) = 0\}.$$

$$\int_{\mathbb{R}^n} j(u^*, |\nabla u^*|)dx = \int_{\mathbb{R}^n} j(u, |\nabla u|)dx, \quad \mathcal{L}^n(C^* \cap (u^*)^{-1}(0, M)) = 0$$

and strict convexity of $\{\xi \mapsto j(s, |\xi|)\}$ *imply that*

$$u(x) = u^*(x - x_0), \quad x \in \mathbb{R}^n.$$

for some $x_0 \in \mathbb{R}^N$

allowing **any** minimizer of some variational pb to be radial. Idea: reducing to $\|\nabla u^*\|_p = \|\nabla u\|_p$ and thus to the framework of Brothers-Ziemer.

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Other directions of work:

As an application of the symmetrization inequalities (including identity cases) we study the following general minimisation problem

$$T = \inf \left\{ J(u) : \ u \in \mathcal{C} \right\},$$

where

$$\mathcal{C} = \left\{ u \in W^{1,p} : G_k(u_k), j_k(u_k, |\nabla u_k|) \in L^1, \ \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k) dx = 1 \right\},$$

where J is the functional defined, for $u = (u_1, \dots, u_m)$, by

$$J(u) = \sum_{k=1}^{m} \int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|) dx - \int_{\mathbb{R}^N} F(|x|, u_1, \dots, u_m) dx.$$

Suitable assumption on j_k, F. Existence, symmetry of minimizers.

Quasi-linear Schrödinger equations (plasma physics, quantum mechanics)

We study the quasi-linear Schrödinger equation

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

By standing waves, we mean solutions of the form $\phi_\omega(t,x) = u_\omega(x)e^{-i\omega t}$. Here ω is a fixed parameter and $\phi_\omega(t,x)$ satisfies the problem if and only if u_ω is a solution of the equation

$$-\Delta u - u\Delta|u|^2 + \omega u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^N.$$

Quasi-linear Schrödinger equations (plasma physics, quantum mechanics)

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$$\begin{cases} i\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

By standing waves, we mean solutions of the form $\phi_\omega(t,x) = u_\omega(x)e^{-i\omega t}$.
Here ω is a fixed parameter and $\phi_\omega(t,x)$ satisfies the problem if and only if u_ω is a solution of the equation

$$-\Delta u - u\Delta|u|^2 + \omega u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^N.$$

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Ground states

We say that a weak solution of the problem is a ground state if it holds $\mathcal{E}_\omega(u) = m_\omega$, where

$$m_\omega = \inf\{\mathcal{E}_\omega(u) : u \text{ is a nontrivial weak solution}\}.$$

Here, $\mathcal{E}_\omega$ is the action associated and reads

$$\mathcal{E}_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla |u|^2|^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx.$$

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Theorem (**Behaviour of ground states**)

For all $\omega > 0$, $\mathcal{G}_\omega$ is non void and any $u \in \mathcal{G}_\omega$ is of the form

$$u(x) = e^{i\theta}|u(x)|, \quad x \in \mathbb{R}^N,$$

for some $\theta \in \mathbb{S}^1$. In particular, the elements of $\mathcal{G}_\omega$ are, up to a constant complex phase, real-valued and non-negative. Furthermore **any** *real non-negative ground state $u \in \mathcal{G}_\omega$ satisfies the following properties*

i) $u \in C^2(\mathbb{R}^N)$ *and* $u > 0$ *in* $\mathbb{R}^N$,

ii) *u is radially symmetric and decreasing,*

iii) *for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq 2$, there exists $(c_\alpha, \delta_\alpha) \in (\mathbb{R}^*_+)^2$ such that*

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha |x|}, \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, for $N = 1$, there exists a unique positive ground state.

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Orbital instability

Theorem (Orbital instability)

Assume that $\omega > 0$,

$$p > 3 + \frac{4}{N}.$$

Let $u \in X_{\mathbb{C}}$ *be a ground state solution of*

$$-\Delta u + u\Delta|u|^2 + \omega u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N. \tag{1}$$

Then, for all $\varepsilon > 0$, *there is* $a_0 \in H^{s+2}(\mathbb{R}^N)$ *such that* $\|a_0 - u\|_{H^1(\mathbb{R}^N)} < \varepsilon$ *and the solution* $\phi(t)$ *of the Schrödinger equation with* $\phi(0) = a_0$ *blows up in finite time.*

We establish a virial type identity. Then we introduce some sets invariant under the flow. Then, by a constrained approach, playing between various characterization of the ground states we derive the blow up result without solving a minimization problem, in contrast to Cazenave-Lions.

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We consider the stability issue for the minimizers of

$$m(c) = \inf\{\mathcal{E}(u) : u \in X, \ \|u\|_2^2 = c\},$$

$$X = \Big\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx < \infty \Big\},$$

where the energy $\mathcal{E}$ reads as

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla |u|^2|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx.$$

If $p < 3 + \frac{4}{N}$, then $m(c) > -\infty$, for any $c > 0$.
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Denote by $\mathcal{G}(c)$ the set of solutions to $m(c)$.

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Theorem (Orbital stability)

Assume that

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and let $c > 0$ be such that $m(c) < 0$. Then $\mathcal{G}(c)$ is non void and orbitally stable. Furthermore, in the two following cases

i) $1 < p < 1 + \frac{4}{N}$ *and* $c > 0$,

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Bifurcation phenomena

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Assume that $1 + \frac{4}{N} \leq p \leq 3 + \frac{4}{N}$. *Then there exists* $c(p, N) > 0$ *with*

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Thank you very much!